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# WHO'S WHO IN NETWORKS. WANTED: THE KEY PLAYER

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Finite population noncooperative games with linear-quadratic utilities, where each player decides how much action she exerts, can be interpreted as a network game with local payoff complementarities, together with a globally uniform payoff substitutability component and an own-concavity effect. For these games, the Nash equilibrium action of each player is proportional to her *Bonacich centrality* in the network of local complementarities, thus establishing a bridge with the sociology literature on social networks. This Bonacich–Nash linkage implies that aggregate equilibrium increases with network size and density. We then analyze a policy that consists of targeting the *key player*, that is, the player who, once removed, leads to the optimal change in aggregate activity. We provide a geometric characterization of the key player identified with an *intercentrality measure*, which takes into account both a player's centrality and her contribution to the centrality of the others.

KEYWORDS: Social networks, peer effects, centrality measures, policies.

## 1. INTRODUCTION

THE DEPENDENCE OF INDIVIDUAL OUTCOMES on group behavior is often referred to as *peer effects* in the literature. In standard peer effects models, this dependence is homogeneous across members and corresponds to an *average* group influence. Technically, the marginal utility to one person of undertaking an action is a function of the average amount of the action taken by her *peers*. Generative models of peer effects, though, suggest that this intragroup externality is, in fact, heterogeneous across group members and varies across individuals with their level of group exposure.[2] In this paper, we allow for a general pattern of bilateral influences and analyze the resulting dependence of individual outcome on group behavior.

More precisely, consider a finite population of players with linear-quadratic interdependent utility functions. Take the matrix of cross-derivatives in these players' utilities. Our first task is to decompose additively this matrix of cross-effects into an idiosyncratic component, a global interaction component, and

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[2]For instance, when job information flows through friendship links, employment outcomes vary across otherwise identical agents with their location in the network of such links (Calvó-Armengol and Jackson (2004)). Durlauf (2004) offers an exhaustive survey of the theoretical and empirical literature on peer effects.

a local interaction network. The idiosyncratic effect reflects (part of) the concavity of the payoff function in own efforts. The global interaction effect is uniform across all players and reflects a strategic substitutability in efforts across all pairs of players. Finally, the local interaction component reflects a (relative) strategic complementarity in efforts that varies across pairs of players. The population-wide pattern of these local complementarities is well captured by a network. This description allows for a clear view of global and local externalities and their sign for a given general pattern of interdependencies.

Based on this reformulation, the paper provides three main results. First, we relate individual equilibrium outcomes to the players' positions in the network of local interactions. Second, we show that the aggregate equilibrium outcome increases with the density and the size of the local interaction network. Finally, we characterize an optimal network disruption policy that exploits the geometric intricacies of this network structure.

In network games, the payoff interdependence is, at least in part, rooted in the network structure of players' links.[3] In these games, equilibrium strategies that subsume the payoff interdependence in a consistent manner should naturally reflect the players' network embeddedness. When the relative magnitude of global and local externalities for our decomposition of cross-effects scales adequately, our network game has a unique and interior Nash equilibrium that is proportional to the Bonacich network centrality. This measure was proposed nearly two decades ago in sociology by Bonacich (1987), and counts the number of all paths (not just shortest paths) that emanate from a given node, weighted by a decay factor that decreases with the length of these paths.[4] This is intuitively related to the equilibrium behavior, because the paths capture all possible feedback. In our case, the decay factor depends on how others' actions enter into own action's payoff.

The sociology literature on social networks is well established and extremely active (see, in particular, Wasserman and Faust (1994)). One of the focuses of this literature is, precisely, to propose different measures of network centralities and to assert the descriptive and/or prescriptive suitability of each of these measures to different situations.[5] This paper provides a behavioral foundation for the Bonacich index, thus singling it out from the vast catalogue of network measures.

The relationship between equilibrium strategic behavior and network topology given by the Bonacich measure allows for a general comparative statics exercise. We show that a denser and larger network of local interactions increases

[3] Jackson (2006) offers a very complete critical survey of the theoretical literature on the economics of networks.

[4] It was originally interpreted as an index of influence or power of the actors of a social network. Katz (1953) is a seminal reference.

[5] See Borgatti (2003) for a discussion on the lack of a systematic criterion to pick up the "right" network centrality measure for each particular situation.

the aggregate equilibrium outcome. This is roughly because both the number and the weight of network paths increase with the network connections.

When the Nash–Bonacich linkage holds, the variance of equilibrium actions reflects the variance of network centralities. In this case, a planner may want to remove a few suitably selected targets from the local interaction network, so as to alter the whole distribution of outcomes. To characterize the network optimal targets, we propose a new measure of network centrality, the *intercentrality measure*, that does not exist in the social network literature. Players with the highest intercentrality are the *key players*, whose removal results in the maximal decrease in overall activity.

Contrary to the Bonacich centrality measure, this new centrality measure does not derive from strategic (individual) considerations, but from the planner's optimality (collective) concerns. Bonacich centrality fails to internalize all the network payoff externalities agents exert on each other, whereas the intercentrality measure internalizes them all. Indeed, removing a player from a network has two effects. First, less players contribute to the aggregate activity level (direct effect) and, second, the network topology is modified, the remaining players thus adopting different actions (indirect effect). As such, the intercentrality measure accounts not only for individual Bonacich centralities, but also for cross-contributions across them. In particular, the key player is not necessarily the player with the highest equilibrium outcome.

Section 2 presents the model and Section 3 describes the Bonacich centrality measure. Section 4 contains the equilibrium analysis and Section 5 explicates the network-based policy. Section 6 discusses a number of extensions. All proofs are relegated to the Appendix.

## 2. THE GAME

Each player $i = 1, \dots, n$ selects an effort $x_i \geq 0$ and obtains the bilinear payoff

$$u_i(x_1, \dots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j, \tag{1}$$

which is strictly concave in own effort, that is, $\partial^2 u_i / \partial x_i^2 = \sigma_{ii} < 0$. We set $\alpha_i = \alpha > 0$ and $\sigma_{ii} = \sigma$ for all $i = 1, \dots, n$. Net of bilateral influences, players have the same payoffs. Remarks 1 and 2 examine the case with general $\alpha_i$'s and $\sigma_{ii}$'s. Bilateral influences are captured by the cross-derivatives $\partial^2 u_i / \partial x_i \partial x_j = \sigma_{ij}$, $i \neq j$, that are pair-dependent and can be of either sign. When $\sigma_{ij} > 0$, an increase in $j$'s effort triggers a upwards shift in $i$'s response. We say that $i$ and $j$ efforts are strategic complements from $i$'s perspective. Reciprocally, when $\sigma_{ij} < 0$, these two efforts are strategic substitutes from $i$'s perspective.

Denote by $\mathbf{\Sigma} = [\sigma_{ij}]$ the square matrix of cross-effects. We use $\mathbf{\Sigma}$ as shorthand for the simultaneous move $n$-player game with payoffs (1) and strategy spaces $\mathbb{R}_+$. We decompose the matrix $\mathbf{\Sigma}$ additively into an idiosyncratic concavity component, a global (uniform) substitutability component, and a local complementarity component.

Let $\underline{\sigma} = \min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = \max\{\sigma_{ij} \mid i \neq j\}$. We assume that $\sigma < \min\{\underline{\sigma}, 0\}$. When $\underline{\sigma} \geq 0$, this is simply the concavity of payoffs in own effort. When $\underline{\sigma} < 0$, this requires that own marginal returns decrease with the level of $x_i$ at least as much as cross-marginal returns do. Let $\gamma = -\min\{\underline{\sigma}, 0\} \geq 0$. If efforts are strategic substitutes for some pair of players, then $\underline{\sigma} < 0$ and $\gamma > 0$; otherwise, $\underline{\sigma} \geq 0$ and $\gamma = 0$. Let $\lambda = \overline{\sigma} + \gamma \geq 0$. We assume that $\lambda > 0$. This is a generic property, because $\lambda = 0$ if and only if $\underline{\sigma} = \overline{\sigma}$, and this equality is not robust to small perturbations of the coefficients $\underline{\sigma}$ and $\overline{\sigma}$.[6] Let $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$ for $i \neq j$ and set $g_{ii} = 0$. By construction, $0 \leq g_{ij} \leq 1$. The parameter $g_{ij}$ measures the relative complementarity in efforts from $i$'s perspective within the pair $(i, j)$ with respect to the benchmark value $-\gamma \leq 0$. This measure is expressed as a fraction of $\lambda$. The matrix $\mathbf{G} = [g_{ij}]$ is a zero-diagonal nonnegative square matrix, interpreted as the adjacency matrix of the network $\mathbf{g}$ of relative payoff complementarities across pairs. When $\sigma_{ij} = \sigma_{ji}$, $\mathbf{G}$ is symmetric and $\mathbf{g}$ is undirected. When $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$ with $\underline{\sigma} \leq 0$, $\mathbf{G}$ is a symmetric $(0, 1)$ matrix, and $\mathbf{g}$ is undirected and unweighted; $\mathbf{g}$ can then be represented by a graph without loops or multiple links.[7] Finally, let $\sigma = -\beta - \gamma$, where $\beta > 0$. Given that $\sigma < \min\{\underline{\sigma}, 0\}$, this is without loss of generality.

If $\mathbf{I}$ denotes the $n$-square identity matrix and $\mathbf{U}$ denotes the $n$-square matrix of ones, it is readily checked that

$$\mathbf{\Sigma} = -\beta\mathbf{I} - \gamma\mathbf{U} + \lambda\mathbf{G}. \tag{2}$$

Bilateral influences result from the combination of an idiosyncratic effect, a global interaction, and a local interaction. The idiosyncratic effect reflects (part of) the concavity of payoffs in own effort $-\beta\mathbf{I}$. The global interaction effect corresponds to a uniform substitutability in efforts across all pairs of players $-\gamma\mathbf{U}$. The local interaction effect corresponds to a (relative) complementarity in efforts, potentially heterogeneous across different pairs of players $\lambda\mathbf{G}$. Following the decomposition of $\mathbf{\Sigma}$ in (2), we rewrite payoffs (1) as

$$u_i(x_1, \ldots, x_n) = \alpha x_i - \frac{1}{2}(\beta - \gamma)x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j \quad \text{for all } i = 1, \ldots, n. \tag{3}$$

[6]The set of parameters $\sigma_{ij}$ for which $\underline{\sigma} = \overline{\sigma}$ has Lebesgue measure zero in $\mathbb{R}^{n(n-1)}$.

[7]A loop is a single direct link that starts at $i$ and ends at $i$, that is, $g_{ii} = 1$. A direct link between $i$ and $j$ is multiple when $g_{ij} \in \{2, 3, \ldots\}$. The matrix $\mathbf{G}$ is zero diagonal, thus ruling out loops. In addition, $0 \leq g_{ij} \leq 1$ by construction, thus ruling out multiple links. It is important to note, though, that our results also hold for networks with loops and/or multiple links, but the economic intuitions are less appealing in this case.

Let $\lambda^* = \lambda/\beta$ denote the strength of local interactions relative to own concavity.

## 3. THE BONACICH NETWORK CENTRALITY MEASURE

Before turning to the equilibrium analysis of the game $\mathbf{\Sigma}$, we define a network centrality measure put forth by Bonacich (1987) that is useful for our analysis. The $n$-square adjacency matrix $\mathbf{G}$ of a network $\mathbf{g}$ keeps track of the direct connections in this network. By definition, $i$ and $j$ are directly connected in $\mathbf{g}$ if and only if $g_{ij} > 0$, in which case $0 \leq g_{ij} \leq 1$ measures the weight associated to this direct connection. Let $\mathbf{G}^k$ be the $k$th power of $\mathbf{G}$, with coefficients $g_{ij}^{[k]}$, where $k$ is some integer. The matrix $\mathbf{G}^k$ keeps track of the indirect connections in the network: $g_{ij}^{[k]} \geq 0$ measures the number of paths of length $k \geq 1$ in $\mathbf{g}$ from $i$ to $j$.[8] In particular, $\mathbf{G}^0 = \mathbf{I}$.

Given a scalar $a \geq 0$ and a network $\mathbf{g}$, we define the matrix

$$\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

These expressions are all well defined for low enough values of $a$.[9] The parameter $a$ is a decay factor that scales down the relative weight of longer paths. If $\mathbf{M}(\mathbf{g}, a)$ is a nonnegative matrix, its coefficients $m_{ij}(\mathbf{g}, a) = \sum_{k=0}^{+\infty} a^k g_{ij}^{[k]}$ count the number of paths in $\mathbf{g}$ that start at $i$ and end at $j$, where paths of length $k$ are weighted by $a^k$. Let $\mathbf{1}$ be the $n$-dimensional vector of ones.

DEFINITION 1: *Consider a network* $\mathbf{g}$ *with adjacency* $n$*-square matrix* $\mathbf{G}$ *and a scalar* $a$ *such that* $\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ *is well defined and nonnegative. The vector of Bonacich centralities of parameter* $a$ *in* $\mathbf{g}$ *is* $\mathbf{b}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1}$.

The Bonacich centrality of node $i$ is $b_i(\mathbf{g}, a) = \sum_{j=1}^{n} m_{ij}(\mathbf{g}, a)$ and it counts the *total* number of paths in $\mathbf{g}$ that start at $i$.[10] It is the sum of all loops $m_{ii}(\mathbf{g}, a)$

[8] A *path* of length $k$ from $i$ to $j$ is a sequence $\langle i_0, \ldots, i_k \rangle$ of players such that $i_0 = i$, $i_k = j$, $i_p \neq i_{p+1}$, and $g_{i_p i_{p+1}} > 0$ for all $0 \leq k \leq k-1$, that is, players $i_p$ and $i_{p+1}$ are directly linked in $\mathbf{g}$. In fact, $g_{ij}^{[k]}$ accounts for the total weight of all paths of length $k$ from $i$ to $j$. When the network is unweighted, that is, $\mathbf{G}$ is a $(0, 1)$ matrix, $g_{ij}^{[k]}$ is simply the number of paths of length $k$ from $i$ to $j$.

[9] Take $a$ smaller than the norm of the inverse of the largest eigenvalue of $\mathbf{G}$.

[10] In fact, $\mathbf{b}(\mathbf{g}, a)$ is obtained from Bonacich's (1987) measure by an affine transformation. Bonacich defines the network centrality measure $\mathbf{h}(\mathbf{g}, a, b) = b[\mathbf{I} - a\mathbf{G}]^{-1}\mathbf{G} \cdot \mathbf{1}$. Therefore, $\mathbf{b}(\mathbf{g}, a) = \mathbf{1} + a\mathbf{h}(\mathbf{g}, a, 1) = \mathbf{1} + \mathbf{k}(\mathbf{g}, a)$, where $\mathbf{k}(\mathbf{g}, a)$ is an early measure of network status introduced by Katz (1953). See also Guimerà, Díaz-Guilera, Vega-Redondo, Cabrales, and Arenas (2002) for related network centrality measures.

from $i$ to $i$ itself and of all the outer paths $\sum_{j\neq i} m_{ij}(\mathbf{g}, a)$ from $i$ to every other player $j \neq i$, that is,

$$b_i(\mathbf{g}, a) = m_{ii}(\mathbf{g}, a) + \sum_{j\neq i} m_{ij}(\mathbf{g}, a).$$

By definition, $m_{ii}(\mathbf{g}, a) \geq 1$ and thus $b_i(\mathbf{g}, a) \geq 1$, with equality when $a = 0$.

## 4. NASH EQUILIBRIUM AND BONACICH CENTRALITY

Consider a matrix of cross-effects $\mathbf{\Sigma}$ that we decompose following (2). We focus on symmetric matrices such that $\sigma_{ij} = \sigma_{ji}$ for all $j \neq i$. Then the largest eigenvalue $\mu_1(\mathbf{G})$ of $\mathbf{G}$ is well defined, with $\mu_1(\mathbf{G}) > 0$ as long as $\sigma_{ij} \neq 0$, for some $j \neq i$.[11] For all $\mathbf{y} \in \mathbb{R}^n$, $y = y_1 + \cdots + y_n$ is the sum of its coordinates. Let $\lambda^* = \lambda/\beta$.

THEOREM 1: *The matrix* $[\beta\mathbf{I} - \lambda\mathbf{G}]^{-1}$ *is well defined and nonnegative if and only if* $\beta > \lambda\mu_1(\mathbf{G})$. *Then the game* $\mathbf{\Sigma}$ *has a unique Nash equilibrium* $\mathbf{x}^*(\mathbf{\Sigma})$, *which is interior and is given by*

$$\mathbf{x}^*(\mathbf{\Sigma}) = \frac{\alpha}{\beta + \gamma b(\mathbf{g}, \lambda^*)}\mathbf{b}(\mathbf{g}, \lambda^*). \tag{4}$$

When the matrix of cross-effects $\mathbf{\Sigma}$ reduces to $\lambda\mathbf{G}$ (that is, $\beta = \gamma = 0$), positive feedback loops escalate without bound and there exists no Nash equilibrium. If, instead, $\mathbf{\Sigma}$ reduces to $-\beta\mathbf{I} - \gamma\mathbf{U}$ (that is, $\lambda = 0$), the equilibrium is generically unique. The condition in Theorem 1 for existence and uniqueness requires that the parameter for own concavity $\beta$ be high enough to counter the payoff complementarity, measured by $\lambda\mu_1(\mathbf{G})$. The scalar $\lambda$ measures the level of positive cross-effects, while $\mu_1(\mathbf{G})$ captures the population-wide pattern of these positive cross-effects. Note that the eigenvalue condition $\beta > \lambda\mu_1(\mathbf{G})$ does not bound the absolute values for these cross-effects, but only their relative magnitude.

The Bonacich-equilibrium expression (4) also implies that each player contributes to the aggregate equilibrium outcome in proportion to her network centrality:

$$x_i^*(\mathbf{\Sigma}) = \frac{b_i(\mathbf{g}, \lambda^*)}{b(\mathbf{g}, \lambda^*)} x^*(\mathbf{\Sigma}).$$

The dependence of individual outcomes on group behavior is referred to as *peer effects*. Here, this intragroup externality is not an average influence. It is

[11]Note that $\mathbf{G}$ is symmetric from the symmetry of $\mathbf{\Sigma}$. By the Perron–Frobenius theorem, the eigenvalues of a symmetric matrix $\mathbf{G}$ are all real numbers. Also, the matrix $\mathbf{G}$ with all zeros in the diagonal has a trace equal to zero. Therefore, $\mu_1(\mathbf{G}) > 0$ whenever $\mathbf{G} \neq \mathbf{0}$.

*heterogeneous* across members, with a variance related to the Bonacich network centrality.

REMARK 1: Consider the general game $(\boldsymbol{\Sigma}, \boldsymbol{\alpha})$, where $\alpha_i > 0$ differs across players. Then a variation of (4) holds that uses the *weighted* Bonacich centrality measure $\mathbf{b}_{\boldsymbol{\alpha}}(\mathbf{g}, \lambda^*) = [\mathbf{I} - \lambda^* \mathbf{G}]^{-1} \cdot \boldsymbol{\alpha}$.

REMARK 2: Consider the general game $(\boldsymbol{\Sigma}, \boldsymbol{\alpha})$, where both $\sigma_{ii} < 0$ and $\alpha_i > 0$ differ across players. Define $(\widetilde{\boldsymbol{\Sigma}}, \widetilde{\boldsymbol{\alpha}})$ by setting $\widetilde{\alpha}_i = \alpha_i/|\sigma_{ii}|$ and $\widetilde{\sigma}_{ij} = \sigma_{ij}/|\sigma_{ii}|$ for all $i, j = 1, \dots, n$. Then the Nash equilibrium for $(\boldsymbol{\Sigma}, \boldsymbol{\alpha})$ corresponds to a variation of (4) that uses the weighted Bonacich centrality of Remark 1 computed for $(\widetilde{\boldsymbol{\Sigma}}, \widetilde{\boldsymbol{\alpha}})$.

REMARK 3: In the proof of Theorem 1, the symmetry of $\boldsymbol{\Sigma}$ does not play any explicit role except to simplify the exposition. In fact, the Bonacich–Nash linkage holds for any asymmetric matrix $\boldsymbol{\Sigma}$ of cross-effects under the condition $\beta > \lambda\rho(\mathbf{G})$, where $\rho(\mathbf{G})$ is the spectral radius of $\mathbf{G}$.

When $\mathbf{G}$ is a symmetric $(0, 1)$ matrix, Theorem 1 holds under a (stronger) sufficient condition that dispenses with computing its largest eigenvalue.

COROLLARY 1: *Suppose that $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$ with $\underline{\sigma} \leq 0$ and that the network $\mathbf{g}$ induced by the decomposition of $\boldsymbol{\Sigma}$ in* (2) *is connected. Let $g = \sum_{i,j} g_{ij}$, which is twice the number of direct links in $\mathbf{g}$. If $\beta > \lambda\sqrt{g + n - 1}$, the only Nash equilibrium of the game $\boldsymbol{\Sigma}$ is given by* (4).

The previous results relate individual equilibrium outcomes to the Bonacich centrality in the network $\mathbf{g}$ of local complementarities. The next result establishes a positive relationship between the aggregate equilibrium outcome and the pattern of local complementarities. For any two matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$, we write $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$ if $\sigma'_{ij} \geq \sigma_{ij}$ for all $i, j$, with at least one strict inequality.

THEOREM 2: *Let $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$ be symmetric such that $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$. If $\beta > \lambda\mu_1(\mathbf{G})$ and $\beta' > \lambda'\mu_1(\mathbf{G}')$ for the decompositions* (2) *of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}'$, then $x^*(\boldsymbol{\Sigma}') > x^*(\boldsymbol{\Sigma})$.*

An increase in the cross-effects from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$ can have opposite effects. First, $\lambda^*\mathbf{G}$ can increase if $\lambda$ and/or $\mathbf{G}$ increase, if $\beta$ decreases or both. In this case, players reap more complementarities in $\boldsymbol{\Sigma}'$ than in $\boldsymbol{\Sigma}$. The number and/or weight of network paths increases, and so does the equilibrium outcome for each player. Second, both $\lambda^*\mathbf{G}$ and $\gamma$ can decrease. This is the case, for instance, if $\boldsymbol{\Sigma}'$ is obtained from $\boldsymbol{\Sigma}$ by only increasing $\underline{\sigma}(\boldsymbol{\Sigma})$. The total number of weighted paths $b(\mathbf{g}, \lambda^*)$ then decreases, but the impact of $\gamma$ dominates, so that the aggregate equilibrium outcome increases.

REMARK 4: When the decompositions (2) of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are such that $(\alpha, \beta, \gamma, \lambda) = (\alpha', \beta', \gamma', \lambda')$ and $\mathbf{G}' \gneq \mathbf{G}$, then $\beta' > \lambda' \mu_1(\mathbf{G}')$ implies that $\beta > \lambda \mu_1(\mathbf{G})$.[12]

## 5. THE KEY PLAYER: A NETWORK-BASED POLICY

In our model, individual equilibrium behavior is tightly rooted in a network structure through (4). We provide a simple geometric criterion to identify the optimal target in the population when the planner wishes to reduce (or to increase) optimally the aggregate group outcome.[13]

We suppose that $\mathbf{\Sigma}$ is symmetric, with $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$, and $\underline{\sigma} \leq 0$. The decomposition of $\mathbf{\Sigma}$ in (2) yields a $(0, 1)$ adjacency matrix $\mathbf{G}$ and an unweighted and undirected network $\mathbf{g}$. Withdraw some player $i$ from the game. We suppose that, for all $v \in \{\underline{\sigma}, \overline{\sigma}\}$, $\sigma_{ij} = \sigma_{i'j'} = v$ for at least two different pairs of players $(i, j)$ and $(i', j')$, differing two by two. This guarantees that any such player removal does not change the values of $\beta$, $\gamma$, and $\lambda$ in the decomposition (2) of $\mathbf{\Sigma}$. We denote by $\mathbf{G}^{-i}$ (resp. $\mathbf{\Sigma}^{-i}$) the new adjacency matrix (resp. matrix of cross-effects), obtained from $\mathbf{G}$ (resp. from $\mathbf{\Sigma}$) by setting to zero all of its $i$th row and column coefficients. The resulting network is $\mathbf{g}^{-i}$.[14]

The planner's problem is to reduce $x^*(\mathbf{\Sigma})$ optimally by picking the appropriate player from the population.[15] Formally, she solves $\max\{x^*(\mathbf{\Sigma}) - x^*(\mathbf{\Sigma}^{-i}) \mid i = 1, \dots, n\}$, which is equivalent to

$$\min\{x^*(\mathbf{\Sigma}^{-i}) \mid i = 1, \dots, n\}. \tag{5}$$

This is a finite optimization problem, that admits at least one solution. Let $i^*$ be a solution to (5). We call $i^*$ the *key player*. Removing $i^*$ from the initial network $\mathbf{g}$ has the highest overall impact on the aggregate equilibrium level. We characterize $i^*$ geometrically.

[12]From the monotonicity of the largest eigenvalue with the coefficients of the matrix in Theorem I* in Debreu and Herstein (1953, p. 600).

[13]Bollobás and Riordan (2003) contains a mathematical analysis of the relative network disruption effects of a topological attack versus random failures in large networks. See also Albert, Jeong, and Barabási (2000) for a numerical analysis for the case of the World Wide Web.

[14]If the primitive of our model is the bilinear expression for the payoffs in (1), the key player analysis applies to any symmetric matrix of cross-effects $\mathbf{\Sigma}$ such that $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i$, $j$, and $\underline{\sigma} \leq 0$; in this case, the network matrix $\mathbf{G}$ in (2) is a $(0, 1)$ matrix. If, instead, the primitive of our model is the expression for the payoffs in (3), the key player analysis carries over to any symmetric adjacency matrix $\mathbf{G}$ with $0 \leq g_{ij} \leq 1$.

[15]Corollary 1 considers the symmetric case where the planner wishes to increase $x^*(\mathbf{g}, \lambda^*)$ optimally.

DEFINITION 2: Consider a network $\mathbf{g}$ with adjacency matrix $\mathbf{G}$ and a scalar $a$ such that $\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ is well defined and nonnegative. The intercentrality of player $i$ of parameter $a$ in $\mathbf{g}$ is

$$c_i(\mathbf{g}, a) = \frac{b_i(\mathbf{g}, a)^2}{m_{ii}(\mathbf{g}, a)}.$$

The Bonacich centrality of player $i$ counts the number of paths in $\mathbf{g}$ that stem from $i$. The intercentrality counts the total number of such paths that hit $i$; it is the sum of $i$'s Bonacich centrality and $i$'s contribution to every other player's Bonacich centrality. Holding $b_i(\mathbf{g}, a)$ fixed, $c_i(\mathbf{g}, a)$ decreases with the proportion of $i$'s Bonacich centrality due to self-loops $m_{ii}(\mathbf{g}, a)/b_i(\mathbf{g}, a)$.

THEOREM 3: *If* $\beta > \lambda\mu_1(\mathbf{G})$, *the key player* $i^*$ *who solves* $\min\{x^*(\mathbf{\Sigma}^{-i}) \mid i = 1, \ldots, n\}$ *has the highest intercentrality of parameter* $\lambda^*$ *in* $\mathbf{g}$, *that is*, $c_{i^*}(\mathbf{g}, \lambda^*) \geq c_i(\mathbf{g}, \lambda^*)$ *for all* $i = 1, \ldots, n$.

The proof for this result uses the following identity, which characterizes all the path changes in a network when a node is removed.

LEMMA 1: *Let* $\mathbf{M}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$ *be well defined and nonnegative. Then* $m_{ij}(\mathbf{g}, a)m_{ik}(\mathbf{g}, a) = m_{ii}(\mathbf{g}, a)[m_{jk}(\mathbf{g}, a) - m_{jk}(\mathbf{g}^{-i}, a)]$ *for all* $k \neq i \neq j$.

EXAMPLE 1: Consider the network $\mathbf{g}$ in Figure 1 with $n = 11$ players. Player 1 bridges together two fully intraconnected groups with five players each. Removing player 1 disrupts the network; removing player 2 decreases maximally the total number of network links.

Table I gives the Bonacich and intercentrality measures for two values of $a$. An asterisk identifies the highest column value.[16]

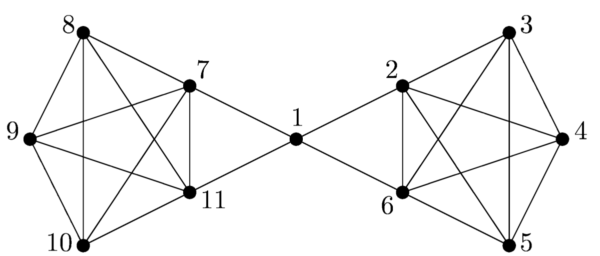


FIGURE 1.

[16]Here, the highest value for $\lambda^*$ compatible with our definition of centrality measures is $\frac{2}{3+\sqrt{41}} \simeq 0.213$.

TABLE I

| | $a = 0.1$ | | $a = 0.2$ | |
|---|---|---|---|---|
| Player Type | $b_i$ | $c_i$ | $b_i$ | $c_i$ |
| 1 | 1.75 | 2.92 | 8.33 | 41.67* |
| 2 | 1.88* | 3.28* | 9.17* | 40.33 |
| 3 | 1.72 | 2.79 | 7.78 | 32.67 |

Player 2 has the highest number of direct links and a wide span of indirect links through her link with player 1. Player 2 has the highest Bonacich centrality. When $a$ is low, player 2 is also the key player. When $a$ is high, the most central player is not the key player. Now indirect effects matter, and removing player 1 has the highest joint direct and indirect effect on aggregate outcome.

COROLLARY 2: *If $\beta > \lambda\mu_1(\mathbf{G})$, the key player $i^*$ who solves* $\max\{x^*(\boldsymbol{\Sigma}^{-i}) \mid i = 1, \ldots, n\}$ *has the lowest intercentrality of parameter $\lambda^*$ in $\mathbf{g}$, that is, $c_{i^*}(\mathbf{g}, \lambda^*) \leq c_i(\mathbf{g}, \lambda^*)$ for all $i = 1, \ldots, n$.*

REMARK 5: When $\boldsymbol{\Sigma}$ is not symmetric, Theorem 3 and Corollary 1 still hold, where the intercentrality measure is now given by $\widetilde{c}_i(\mathbf{g}, a) = b_i(\mathbf{g}, a) \times (\sum_{j=1}^{n} m_{ji}(\mathbf{g}, a))/m_{ii}(\mathbf{g}, a)$.

## 6. DISCUSSION AND EXTENSIONS

We discuss a number of possible extensions of this work. First, our analysis is restricted to linear-quadratic utilities that capture linear externalities in players' actions. First-order conditions for interior equilibria then produce a system of linear equations that leads to the Bonacich–Nash linkage. Albeit special, linear-quadratic payoffs are commonly used in a number of economic models, including conformist behavior, crime networks, and research and development agreements in oligopoly.[17] Consider now general $C^2$ utility functions $\mathbf{u}$ that correspond to nonlinear externalities. Let $\boldsymbol{\Sigma}(\mathbf{x}^*)$ be the (symmetric) Jacobian of $\nabla\mathbf{u}$ at an interior Nash equilibrium $\mathbf{x}^* > \mathbf{0}$. Decompose $\boldsymbol{\Sigma}(\mathbf{x}^*)$ as in (2). By a simple continuity argument, the first-order approximation of $\mathbf{x}^*$ corresponds to the Bonacich centrality vector for this decomposition.

Second, Theorem 3 characterizes the key player when the planner's objective function is the aggregate group outcome $x^*(\boldsymbol{\Sigma})$. Suppose, instead, that the planner's objective is to maximize welfare $W^*(\boldsymbol{\Sigma}) = \sum_{i=1}^{n} u_i(\mathbf{x}^*(\boldsymbol{\Sigma})) = (\beta + \gamma)/2 \sum_{i=1}^{n} x_i^*(\boldsymbol{\Sigma})^2$. When $\gamma = 0$, this becomes $2\beta W^*(\boldsymbol{\Sigma}) = \alpha^2 \sum_{i=1}^{n} b_i(\mathbf{g}, \lambda^*)^2$.

[17]See, e.g., Akerlof (1997) and Bernheim (1994) for a model of conformism, Calvó-Armengol and Zenou (2004) for a model of crime networks, and Goyal and Moraga (2001) for a Cournot model with research and development cost-reducing bilateral agreements.

A geometric characterization of the key player is also possible in this case. The building block is provided by Lemma 1.

Third, Theorem 3 geometrically characterizes single player targets, but the intercentrality measure can be generalized to a group index. It is important to note that this group target selection problem is not amenable to a sequential key player problem. For instance, the key group of size 2 in Example 1 when $a = 0.2$ is $\{2, 7\}$, rather than the sequential optimal pair $\{1, 2\}$. In fact, optimal group targets belong to a wider class of problems—the maximization of submodular set functions—that cannot admit exact sequential solutions.[18]

Fourth, beyond the optimal player removal problem, the network policy analysis can also accommodate more general targeted tax/subsidy policies. Take a population of $n+1$ agents $i = 0, 1, \ldots, n$ and a matrix of cross-effects $\mathbf{\Sigma}$ with associated network $\mathbf{g}$ in (2). Suppose that the planner holds $x_0$ to some fixed exogenous value $s \in \mathbb{R}$. The case $s > 0$ (resp. $s < 0$) is a subsidy (resp. tax), while $s = 0$ corresponds to the key player problem solved before. Players $i = 1, \ldots, n$ then play an $n$-player game with interior Bonacich–Nash equilibrium $\mathbf{x}^*_{-0}(\mathbf{\Sigma}^{-0}, s)$. Denote by $\mathbf{g}_0$ the $n$-dimensional column vector with coordinates $g_{01}, \ldots, g_{0n}$ that keeps track of player 0's direct contacts in $\mathbf{g}$. Let $\alpha^* = \alpha/\beta$ and $\gamma^* = \gamma/\beta$. Then the total equilibrium population outcome is $s + x^*_{-0}(\mathbf{\Sigma}^{-0}, s)$, where

$$x^*_{-0}(\mathbf{\Sigma}^{-0}, s) = \frac{1}{1 + \gamma^* b(\mathbf{g}^{-0}, \lambda^*)} \times [(\alpha^* - \gamma^* s) b(\mathbf{g}^{-0}, \lambda^*) + \lambda^* s b_{\mathbf{g}_0}(\mathbf{g}^{-0}, \lambda^*)].$$

Here, $b_{\mathbf{g}_0}(\mathbf{g}^{-0}, \lambda^*)$ is the aggregate weighted Bonacich centrality defined in Remark 1. Given an objective function related to the total population output $s + x^*_{-0}(\mathbf{\Sigma}^{-0}, s)$ and given a set of constraints, the planner's problem is to fix optimally the value of $s$ and the target identity $i$. Holding $s$ constant, the choice of the optimal target is a simple finite optimization problem.

Finally, the analysis so far deals with a fixed network. When $\mathbf{G}$ is a $(0, 1)$ matrix, we can easily endogenize the network with a two-stage game. In the first stage, players decide simultaneously to stay in the network or to drop out of it (and get their outside option). This is a simple binary decision. In the second stage, the players who stay play the network game on the resulting network. Uniqueness of the second-stage Nash equilibrium and its closed-form expression crucially simplify the analysis of this two-stage game. See, e.g., Ballester, Calvó-Armengol, and Zenou (2004) and Calvó-Armengol and Jackson (2004) for analysis along this vein.

[18] See Ballester, Calvó-Armengol, and Zenou (2004).

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## APPENDIX

PROOF OF THEOREM 1: The necessary and sufficient condition for $[\beta\mathbf{I} - \lambda\mathbf{G}]^{-1}$ to be well defined and nonnegative derives from Theorem III* in Debreu and Herstein (1953, p. 601). An interior Nash equilibrium in pure strategies $\mathbf{x}^* \in \mathbb{R}^n_+$ is such that $\partial u_i/\partial x_i(\mathbf{x}^*) = 0$ and $x_i^* > 0$ for all $i = 1, \ldots, n$. If such an equilibrium exists, it then solves

$$-\boldsymbol{\Sigma} \cdot \mathbf{x} = [\beta\mathbf{I} + \gamma\mathbf{U} - \lambda\mathbf{G}] \cdot \mathbf{x} = \alpha\mathbf{1}. \tag{6}$$

The matrix $\beta\mathbf{I} + \gamma\mathbf{U} - \lambda\mathbf{G}$ is generically nonsingular, and (6) has a unique generic solution in $\mathbb{R}^n$, denoted by $\mathbf{x}^*$.[19] Using the fact that $\mathbf{U} \cdot \mathbf{x}^* = x^*\mathbf{1}$, (6) is equivalent to $\beta[\mathbf{I} - \lambda^*\mathbf{G}] \cdot \mathbf{x}^* = (\alpha - \gamma x^*)\mathbf{1}$. By inverting the matrix, we get $\beta\mathbf{x}^* = (\alpha - \gamma x^*)\mathbf{b}(\mathbf{g}, \lambda^*)$ and (4) follows by simple algebra. Given that $\alpha > 0$ and $b_i(\mathbf{g}, \lambda^*) \geq 1$ for all $i$, it follows that $\mathbf{x}^*$ is interior. This also show that $\mathbf{x}^*$ is the unique interior equilibrium.

To establish uniqueness, we deal with corner solutions. For all matrixes $\mathbf{Y}$, vectors $\mathbf{y}$, and $S \subset \{1, \ldots, n\}$, $\mathbf{Y}_S$ is the submatrix of $\mathbf{Y}$ with rows and columns in $S$, and $\mathbf{y}_S$ is the subvector of $\mathbf{y}$ with rows in $S$.

Recall that $\beta, \gamma$, and $\lambda$ are the original parameters from the decomposition (2) of $\boldsymbol{\Sigma}$. For all $S \subset \{1, \ldots, n\}$, we can write the matrix $\Sigma_S$ as a function of these parameters, i.e., $\Sigma_S = \beta\mathbf{I}_S + \gamma\mathbf{U}_S - \lambda\mathbf{G}_S$, so that $\mathbf{G}_S$ is the adjacency matrix of the subnetwork induced by the players in $S$. From Theorem I* in Debreu and Herstein (1953, p. 600), it is easy to see that $\mu_1(\mathbf{G}) \geq \mu_1(\mathbf{G}_S)$ and thus $\beta > \lambda\mu_1(\mathbf{G}_S)$.

Let $\mathbf{y}^*$ be a noninterior Nash equilibrium of the game $\boldsymbol{\Sigma}$. Let $S \subset \{1, \ldots, n\}$ such that $y_i^* = 0$ if and only if $i \in N \backslash S$; $y_i^* > 0$ for all $i \in S$. Note that $S \neq \emptyset$,

[19]The set of parameters $\beta$, $\gamma$, and $\lambda$ for which $\det(\beta\mathbf{I} + \gamma\mathbf{U} - \lambda\mathbf{G}) = 0$ has Lebesgue measure zero in $\mathbb{R}^3$.

because $\partial u_i/\partial x_i(\mathbf{0}) = \alpha > 0$, and $\mathbf{0}$ cannot be a Nash equilibrium. Then $-\boldsymbol{\Sigma}_S \cdot \mathbf{y}_S^* = [\beta \mathbf{I}_S + \gamma \mathbf{U}_S - \lambda \mathbf{G}_S] \cdot \mathbf{y}_S^* = \alpha \mathbf{1}_S$. Given that $\beta > \lambda \mu_1(\mathbf{G}_S)$, we have

$$\mathbf{y}_S^* = \frac{\alpha - \gamma y_S^*}{\beta} \mathbf{b}(\mathbf{g}_S, \lambda^*). \tag{7}$$

Now every player $i \in N \backslash S$ is best responding with $y_i^* = 0$, so that

$$\frac{\partial u_i}{\partial x_i}(\mathbf{y}^*) = \alpha + \sum_{j \in S} \sigma_{ij} y_j^* = \alpha - \gamma y_S^* + \lambda \sum_{j \in S} g_{ij} y_j^* \leq 0 \quad \text{for all } i \in N \backslash S,$$

where the last equality uses the decomposition of $\boldsymbol{\Sigma}$. Using (7), we rewrite this inequality as

$$(\alpha - \gamma y_S^*)\left(1 + \lambda^* \sum_{j \in S} g_{ij} b_j(\mathbf{g}_S, \lambda^*)\right) \leq 0,$$

which is equivalent to $\alpha - \gamma y_S^* \leq 0$. Using (7), we conclude that $y_i^* \leq 0$ for all $i \in S$, which is a contradiction. *Q.E.D.*

PROOF OF COROLLARY 1: The proof is obtained from the upper bound for the largest eigenvalue of a connected graph in Theorem 1.5 in Cvetković and Rowlinson (1990, p. 5). *Q.E.D.*

PROOF OF THEOREM 2: Let $\boldsymbol{\Sigma}' \gneq \boldsymbol{\Sigma}$. We write $\boldsymbol{\Sigma}' = \boldsymbol{\Sigma} + \lambda \mathbf{D}$, $d_{ij} \geq 0$, with at least one strict inequality, and write $\lambda$ from (2) for $\boldsymbol{\Sigma}$. If $\beta > \lambda \mu_1(\mathbf{G})$ and $\beta' > \lambda' \mu_1(\mathbf{G}')$, Theorem 1 holds and $-\boldsymbol{\Sigma} \cdot \mathbf{x}^*(\boldsymbol{\Sigma}) = -\boldsymbol{\Sigma}' \cdot \mathbf{x}^*(\boldsymbol{\Sigma}') = \alpha \mathbf{1}$ with $\mathbf{x}^*(\boldsymbol{\Sigma}), \mathbf{x}^*(\boldsymbol{\Sigma}') > \mathbf{0}$. From the symmetry of $\boldsymbol{\Sigma}'$,

$$\alpha x^*(\boldsymbol{\Sigma}') = -\mathbf{x}^{*t}(\boldsymbol{\Sigma}') \cdot \boldsymbol{\Sigma} \cdot \mathbf{x}^*(\boldsymbol{\Sigma}) = \alpha x^*(\boldsymbol{\Sigma}) + \lambda \mathbf{x}^{*t}(\boldsymbol{\Sigma}') \cdot \mathbf{D} \cdot \mathbf{x}^*(\boldsymbol{\Sigma}).$$

Given that $\alpha > 0$, we conclude that $x^*(\boldsymbol{\Sigma}') > x^*(\boldsymbol{\Sigma})$. *Q.E.D.*

PROOF OF LEMMA 1: From $\boldsymbol{\Sigma}$ symmetric, $m_{jk}(\mathbf{g}, a) = m_{kj}(\mathbf{g}, a)$ for all $j, k$, and $\mathbf{g}$. Then

$$\begin{aligned} & m_{ii}(\mathbf{g}, a)[m_{jk}(\mathbf{g}, a) - m_{jk}(\mathbf{g}^{-i}, a)] \\ &\quad = \sum_{p=1}^{+\infty} a^p \sum_{\substack{r+s=p \\ r \geq 0, s \geq 1}} g_{ii}^{[r]} \big(g_{jk}^{[s]} - g_{j(i^0)k}^{[s]}\big) = \sum_{p=1}^{+\infty} a^p \sum_{\substack{r+s=p \\ r \geq 0, s \geq 2}} g_{ii}^{[r]} g_{j(i)k}^{[s]} \\ &\quad = \sum_{p=1}^{+\infty} a^p \sum_{\substack{r'+s'=p \\ r' \geq 1, s' \geq 1}} g_{ji}^{[r']} g_{ik}^{[s']} = m_{ji}(\mathbf{g}, a) m_{ik}(\mathbf{g}, a), \end{aligned}$$

where $g^{[s]}_{j(i^0)k}$ (resp. $g^{[s]}_{j(i)k}$) is the weight of $s$-length paths from $j$ to $k$ that do not contain $i$ (resp. contain $i$) and $g^{[0]}_{ii} = 1$. *Q.E.D.*

PROOF OF THEOREM 3: Note that $\mu_1(\mathbf{G}) \geq \mu_1(\mathbf{G}^{-i})$. Therefore, when $\mathbf{M}(\mathbf{g}, \lambda^*)$ is well defined and nonnegative, so is $\mathbf{M}(\mathbf{g}^{-i}, \lambda^*)$ for all $i = 1, \dots, n$. When $\alpha > 0$, $x^*(\mathbf{\Sigma}^{-i})$ increases in $b(\mathbf{g}^{-i}, \lambda^*)$ and (5) is equivalent to $\min\{b(\mathbf{g}^{-i}, \lambda^*) \mid i = 1, \dots, n\}$. Define $b_{ji}(\mathbf{g}, \lambda^*) = b_j(\mathbf{g}, \lambda^*) - b_j(\mathbf{g}^{-i}, \lambda^*)$ for all $j \neq i$. This is the contribution of $i$ to $j$'s Bonacich centrality in $\mathbf{g}$. Summing over all $j \neq i$ and adding $b_i(\mathbf{g}, \lambda^*)$ to both sides gives

$$b(\mathbf{g}, \lambda^*) - b(\mathbf{g}^{-i}, \lambda^*) = b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} b_{ji}(\mathbf{g}, \lambda^*) \equiv d_i(\mathbf{g}, \lambda^*).$$

The solution of (5) is $i^*$ such that $d_{i^*}(\mathbf{g}, \lambda^*) \geq d_i(\mathbf{g}, \lambda^*)$ for all $i = 1, \dots, n$. We have

$$\begin{aligned} d_i(\mathbf{g}, \lambda^*) &= b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} [b_j(\mathbf{g}, \lambda^*) - b_j(\mathbf{g}^{-i}, \lambda^*)] \\ &= b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} [m_{jk}(\mathbf{g}, \lambda^*) - m_{jk}(\mathbf{g}^{-i}, \lambda^*)]. \end{aligned}$$

Using Lemma 1, this becomes:

$$\begin{aligned} d_i(\mathbf{g}, \lambda^*) &= b_i(\mathbf{g}, \lambda^*) + \sum_{j \neq i} \sum_{k=1}^{n} \frac{m_{ij}(\mathbf{g}, a) m_{ik}(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)} \\ &= b_i(\mathbf{g}, \lambda^*) \left[ 1 + \sum_{j \neq i} \frac{m_{ij}(\mathbf{g}, a)}{m_{ii}(\mathbf{g}, a)} \right] = \frac{b_i(\mathbf{g}, \lambda^*)^2}{m_{ii}(\mathbf{g}, \lambda^*)}. \end{aligned}$$

*Q.E.D.*

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