Market Design: Lecture 5

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Recap

5. b) Many-to-one markets: substitutable preferences
6. a) Large market results: empirical evidence

Outline

6. b) Large market results: incentives, couples

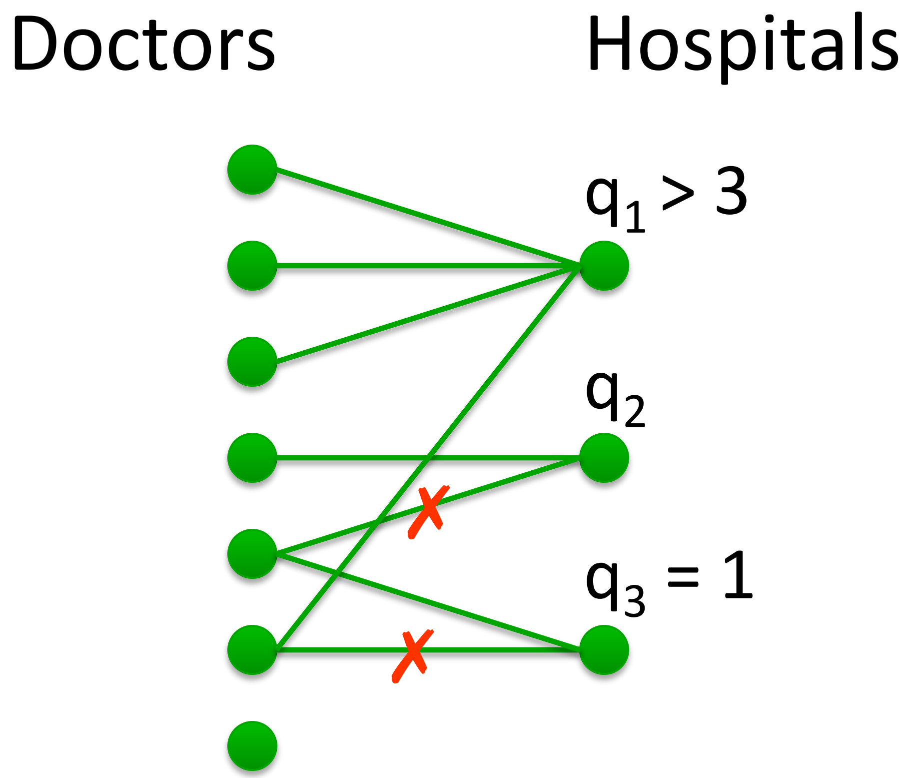
Part 6: Large Market Results.

Theoretical Result

Theorem. Even allowing women *arbitrary* preferences, the fraction of agents with more than one stable mate tends to zero as n tends to infinity (holding k fixed).

[Immorlica-Mahdian '05, Kojima-Pathak '09]

Intuition



One-to-One Markets

Proof Sketch:

1. An algorithm that counts the number of stable husbands of a given woman.
2. Bounding probability of having > 1 stable husband in terms of the number of singles.
3. Bounding the number of singles by the solution of the occupancy problem.

Step 1: Finding Stable Husbands of g

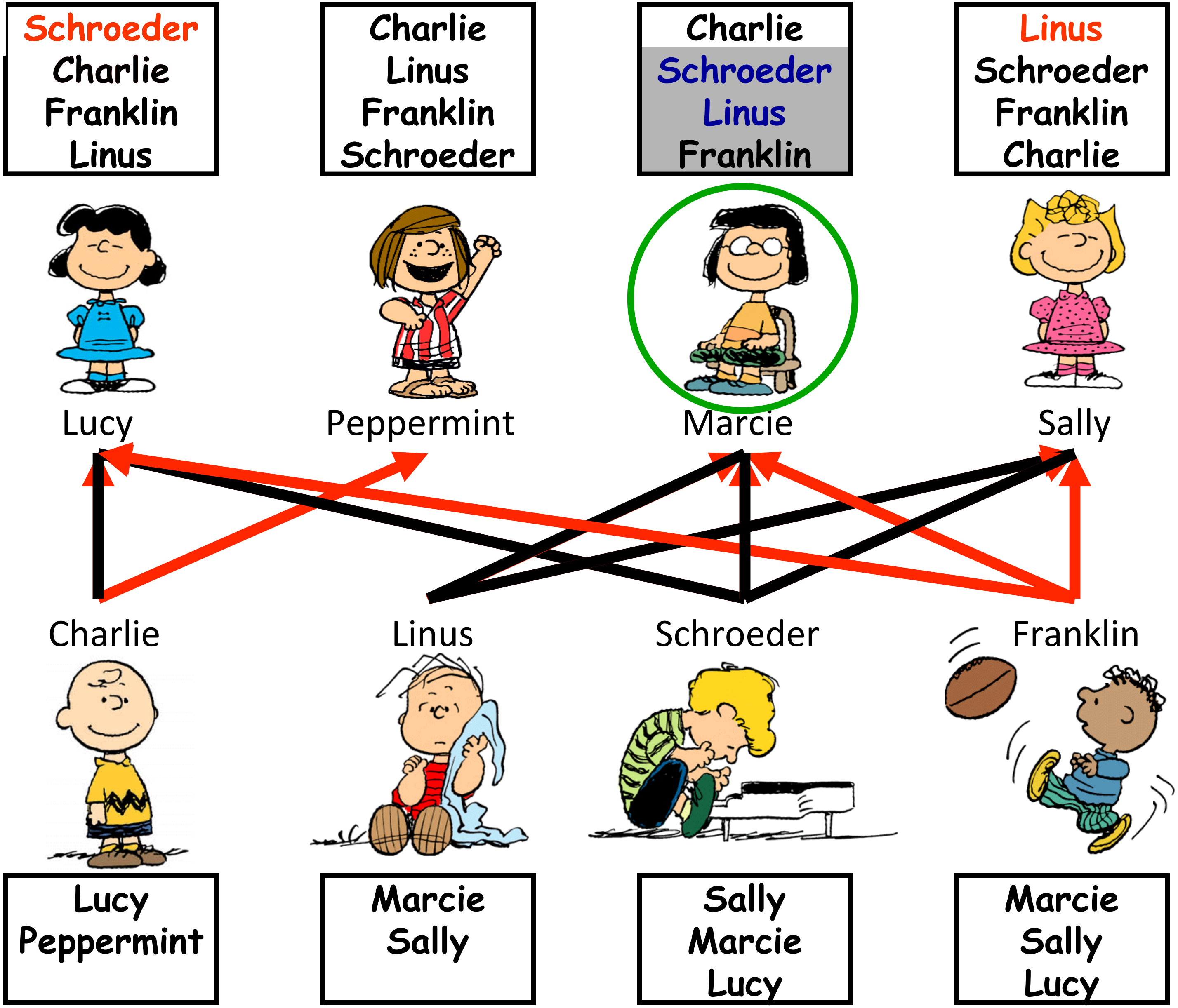
- Use men-proposing to find stable matching
- Whenever algorithm finds stable matching,
  - have g divorce husband and truncate preference
  - continue men-proposing algorithm
- Terminate whenever
  - previously married man runs through his list, or
  - previously single woman receives a proposal

Question. If each woman has an arbitrary complete preference list, and each man has a random list of k women, what is the probability that this algorithm returns more than one stable husband for g?

The main tool that we will use to answer this question is the *principle of deferred decisions*:

> Men do not pick the list of their favorite women in advance; Instead, every time a man needs to propose, he picks a woman at random and proposes to her. A man remains single if he gets rejected by k different women.

End!



Schroeder
Charlie
Franklin
Linus
Charlie
Linus
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Linus
Schroeder
Franklin
Charlie
Lucy
Peppermint
Marcie
Sally
Charlie
Linus
Schroeder
Franklin
Lucy
Peppermint
Marcie
Sally
Sally
Marcie
Lucy
Marcie
Sally
Lucy

Step 2: Bounding the Probability

- Consider first stable matching μ found by alg.
- Let A_μ = {single women in μ}, and $X_\mu = |A_\mu|$.
- Conditioning on random choices made before algorithm finds μ,

  $\Pr[g \text{ has} > 1 \text{ stable mate} \mid \mu] <$

  $\Pr[g \text{ gets another proposal} \mid \mu] = 1 / (X_\mu+1)$
- Removing conditioning, prob. < $E_\mu [1 / (X_\mu+1)]$

Step 3: Number of Singles

- Want to compute $E_\mu[1 / (X_\mu+1)]$
- Note probability woman remains single is at least probability she's never named by a man.
- Let $Y_{m,n}$ = # empty bins when m balls thrown randomly into n bins.

Lemma. $$E_\mu\left(\frac{1}{(X_\mu+1)}\right) \leq E_\mu\left(\frac{1}{(Y_{(k+1)n,n}+1)}\right) + \frac{k^2}{n}$$

The Occupancy Problem

Lemma. $$E_\mu\left(\frac{1}{(Y_{m,n}+1)}\right) \leq \frac{e^{m/n}}{n}$$

Proof Sketch.

- Use the principle of inclusion and exclusion to compute $E[1/(Y_{m,n}+1)]$ as a summation.
- Compare this summation to another (known) summation term-by-term.

Putting it all together…

Theorem. In the model where women have arbitrary complete preference lists and men have random lists of size k, the probability that a fixed woman has more than one stable husband is at most

$$\frac{e^{k+1} + k^2}{n}.$$

Extensions

Arbitrary IID distributions?

Extensions

Women: n hospital positions, preference is a uniform random permutation of all men

Men: n applicants, preference chosen by:

- distribution *D* over women
- construct list iteratively by selecting from *D*

Extension

Theorem. In the above model, the probability that a fixed woman has more than one stable husband is at most:

$$\frac{16k}{\ln(n)} + \frac{3\ln(n)}{4k\sqrt{n}} = O\left(\frac{k}{\ln(n)}\right)$$

Many-to-One Extension

Theorem. Truthfulness is almost surely a best response when others are truthful.

Proof Sketch:

- Argue "dropping strategies" comprehensive
- Modify alg counting stable husbands to study rejection chains of dropping strategies
- Argue rejection chains don't return to college w/high prob. when market large