# Correspondence

## On Constructing Embedded Multilevel Trellis Codes

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*Abstract*—**A design technique is presented to reduce the search time for trellis codes with multilevel/phase modulation. Codes are constructed by connecting trellis diagrams for codes with fewer states in parallel. For example, an $N$-state code can be constructed by connecting two $N/2$-state codes. The way in which the embedded codes are connected increases the upper limit on minimum free distance otherwise imposed by parallel transitions between states. In some cases, this technique can reduce the number of codes in a code search by a factor of approximately $2^\nu$, the number of coder states. A computer search incorporating this technique for eight-level amplitude modulation (8-AM) codes having $2^{11}$ and $2^{12}$ states produced codes with greater minimum free distance than reported previously [1] (i.e., greater than 6 dB coding gain). New eight-level phase shift-keying (8-PSK) codes, which have a different structure from previously reported codes [1], are also presented.**

### I. Introduction

Trellis coding combined with multilevel/phase signaling [1] has received much attention as a means for reducing the signal-to-noise ratio necessary to achieve a desired error rate for high-speed date transmission. So far, the best multilevel trellis codes have been found by computer search. Two optimality criteria, minimum free distance and an upper bound on the decoding error probability, have been used [1], [2]. Because the number of codes over which to search grows very rapidly with the number of states, an exhaustive search for the best code with a given number of states, assuming a specific signal constellation, is often impractical. Design rules, such as those proposed by Ungerboeck [1], are therefore needed to reduce the size of the code space to be searched.

Here we propose an additional design technique which can be used to reduce the search time for codes with a large number of states. The technique consists of connecting trellises in "parallel." A search for eight-level amplitude modulation (8-AM) codes composed of two trellis diagrams connected in parallel produced codes with greater coding gains than those reported in [1]. Codes for phase shift keying with eight different phases (8-PSK), which have a different structure from those reported in [1], can readily be found with the same minimum free distance.

### II. Ungerboeck's Design Rules

A rate $m/(m+1)$ convolutional coder is shown in Fig. 1. The coder outputs $m+1$ bits which are mapped to a point in a signal constellation containing $2^{m+1}$ points. Denote the vector of $m$ source bits at the $T$th iteration as $a_T$, and the vector of $m+1$ coder output bits as $y_T$. The *minimum free distance* is defined as

$$D^2_{\min} = \min_{\{P,P',T\geq 0\}} \sum_{i=0}^{T} d^2(y_i, y_i')$$

where $P$ and $P'$ represent two distinct paths in the trellis, i.e., sequences of source bits, corresponding to a particular error event of length $T$. $d(y_i, y_i')$ is the Euclidean distance between the signal points corresponding to the coder output vectors $y_i$ and $y_i'$, which are associated with paths $P$ and $P'$, respectively. For a given number of coder states, we wish to find the particular code or codes whose $D_{\min}$ is greater than or equal to $D_{\min}$ for any other code.

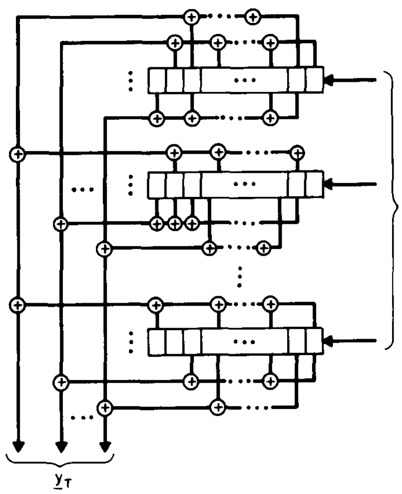


Fig. 1. Rate $m/m+1$ convolutional encoder. There are $m$ shift registers and $m+1$ output lines.

Mapping coder output bits to signal constellation points by set partitioning [1] is illustrated for the 8-AM constellation in Fig. 2. At each level of the binary tree shown in Fig. 2, each constellation is subdivided into two different subconstellations having increased distance between the signal points. The leaves of the tree consist of single signal points. Binary coder outputs are assigned to each signal point as shown.

Design rules proposed by Ungerboeck are as follows [1].

1) Each signal point in the constellation must appear an equal number of times in the trellis.

2) If the maximum value of $D_{\min}$ for codes with a given number of states is known to be less than the distance between constellation points at a given depth in the set-partitioning tree, then the points in each of the constellations at this depth are assigned to parallel transitions between pairs of states. As an example, the best rate 2/3, 8-AM trellis code with $2^6$ states has $D_{\min} < 4\delta$, which is the distance between points in the constellations at depth "$C$" shown in Fig. 2. This code therefore has two parallel transitions between pairs of states corresponding to a particular constellation at depth $C$.

3) Signal points from only one constellation at depth $B$ in the set-partitioning tree are assigned to all transitions out of and in to a particular state.

The first rule imposes a regular structure. The second rule ensures that the trellis has minimum "connectivity," i.e., the length of the minimum distance error event is made large. The third rule maximizes the distance incurred by an incorrect path upon diverging and merging with the transmitted path.

### III. Embedded Codes

The embedding technique is explained by way of an example. Consider the design of a rate 2/3 trellis code

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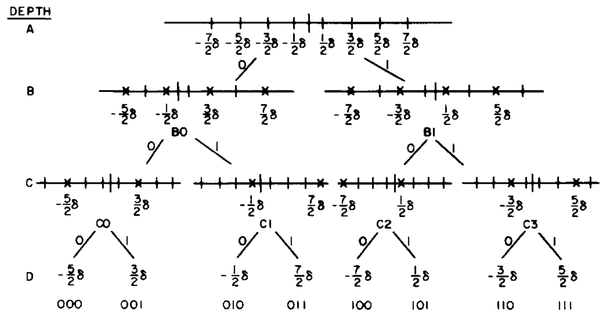


Fig. 2. Set-partitioned 8-AM signal constellation.

combined with 8-level AM. The best codes presented in [1] with $2^\nu$ states where $\nu < 7$ have two parallel transitions between states, so that $D_{\min}$ is bounded by the distance between constellation points at level $C$ in Fig. 2; that is, $D_{\min} \leq 4\delta$ where $\delta$ is the distance between levels (coding gain $\leq$ 5.8 dB). If $\nu \geq 7$, the best codes with parallel transitions have $D_{\min} = 4\delta$. Consequently, $D_{\min}$ for such codes is determined by single-error events. In particular, any path which diverges from the transmitted path and transits to a different state must accumulate a squared Euclidean distance of at least $16\delta^2$. 8-AM codes with $\nu > 7$ and $D_{\min} > 4\delta$ therefore cannot have parallel transitions.

A rate 2/3 code is shown in Fig. 3(a) in which the top bit is the input to a shift register with $\nu - 1$ delays and the bottom bit is the input to a shift register with one delay. The state of the code consists of the right $\nu - 1$ bits in the top shift register plus the right bit in the bottom shift register, making a total of $2^\nu$ states. The trellis diagram for this code consists of two $2^{\nu-1}$-state trellis diagrams, each representing the top shift register, in parallel with transitions between them. The transitions between the top four trellis states (two sets of two states belonging to each embedded trellis) are illustrated in Fig. 3(b). The convolutional coder state is shown to the left of each trellis state. States contained in each of the two embedded $2^{\nu-1}$-state trellis diagrams are labeled "$T1$" and "$T2$," respectively.

The trellis code in Fig. 3(a) ensures that an error event spans at least two symbols. For instance, if the transmitted path corresponds to a source sequence of all zeros, an incorrect path might correspond to a one shifted into the bottom shift register in Fig. 3(a) followed by a zero, and zeros shifted into the top register. The trellis path corresponding to this sequence consists of a transition from trellis $T1$ to $T2$ and a subsequent transition from $T2$ to $T1$. If the diverging path does not merge with the transmitted path after two symbols, however, then the length of the error event must be at least $\nu - 1$.

An assignment of signal points to trellis transitions which guarantees that all error events of length two have a squared distance of at least $20\delta^2$ is shown in Fig. 3(b). Each integer $i$ shown next to each trellis transition in Fig. 3(b) corresponds to the signal level $(i/2)\delta$. The associated convolutional coder is shown in Fig. 3(a). The $j$th states belonging to trellises $T1$ and $T2$, respectively, will be referred to as "parallel states." Referring to Fig. 3(a), parallel states have the same $\nu - 1$ rightmost bits in the top shift register, but have different bits in the bottom right register. Transitions to parallel states from the same state of origin are assigned points from a constellation at depth $C$ in Fig. 2. Similarly, transitions into the same state originating from parallel states are also assigned points from a particular constellation at level $C$. A two-symbol error event having squared distance $20\delta^2$ is shown in Fig. 4.

The number of delays the top source bit must experience in Fig. 3(a) is nearly twice the number of delays which a source bit experiences in the more conventional coder shown in Fig.

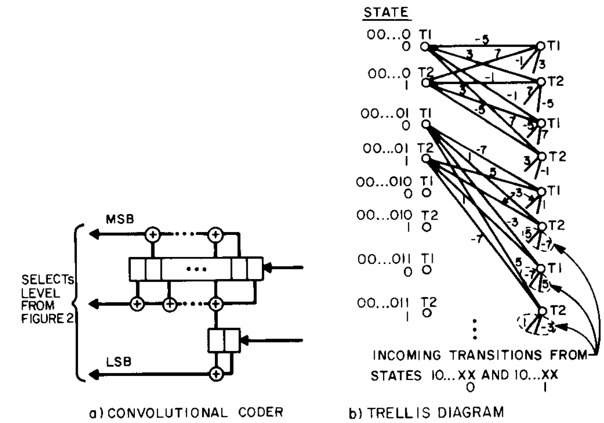


Fig. 3. 8-AM code for which $D^2_{\min} \leq 20\delta^2$. The trellis diagram can be decomposed into two connected trellises "$T1$" and "$T2$."

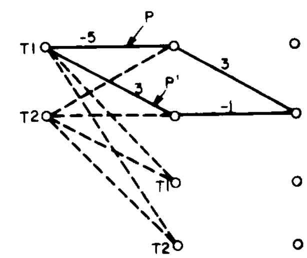


Fig. 4. Two paths $P$ and $P'$ with $d^2(P, P') = 20\delta^2$.

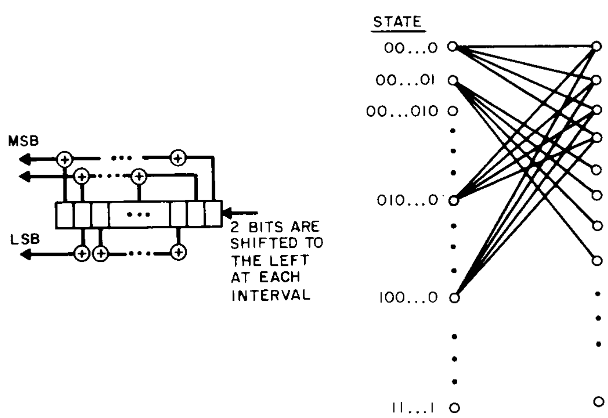


Fig. 5. Conventional rate 2/3 convolutional encoder and trellis diagram.

5, assuming that the number of states is the same in each case. The minimum distance path for the coder in Fig. 3(a) is therefore likely to be longer than the minimum distance path for the coder in Fig. 5. Nevertheless, results from a computer search for the best codes with $\nu$ = 6, 7, and 8 having the structures shown Fig. 3(a) and 5, respectively, indicate that the best codes for each structure have the same $D_{\min}$. Assuming that the number of trellis states is $2^\nu$, the set of all codes with the structure shown in Fig. 3(a) has cardinality $2^{2(\nu-1)}$, whereas the set of all codes with the structure shown in Fig. 5 has cardinality $2^{3(\nu-1)}$. Of course, if we allow the bottom summer in Fig. 3(a) to connect to the top shift register, then the cardinality of both sets is approximately the same. Allowing this extra degree of freedom in the computer search did not, however, uncover any codes with greater $D_{\min}$ than that for the best code with the structure shown in Fig. 3(a) for the cases examined. In each case, the number of codes to examine can be reduced by using rejection rules [1].

$D^2_{\min}$ for the coder shown in Fig. 3(a) must always be less than or equal to $20\delta^2$. To construct a code with $D^2_{\min} > 20\delta^2$, the bottom shift register must have at least two delays, corresponding to four trellis diagrams in parallel. In this case, an error event spanning three symbols should incur a squared distance greater than $20\delta^2$. Fig. 6 shows a coder structure in which such events must incur a squared distance of at least $36\delta^2$.

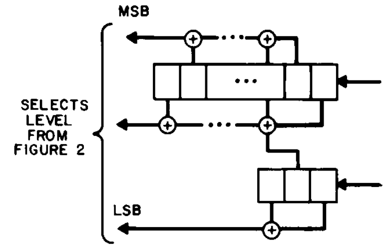


Fig. 6. Rate 2/3, 8-AM trellis code with $D^2_{\min} \leq 36\delta^2$.

## IV. Code Search Results

### A. 8-AM Codes

A computer search for 8-AM trellis codes having the structure shown in Fig. 3(a) was performed for $\nu = 8$ to $\nu = 12$. The codes reported in [1] ($\nu < 8$) have parallel transitions, so that $D_{\min} \leq 4\delta$. No codes with $\nu < 11$ were found having $D_{\min}$ greater than $4\delta$, which is $D_{\min}$ for the $\nu = 7$ code reported in [1]. An exhaustive search for the best code with $2^8$ states was executed for both structures shown in Figs. 5 and 3(a). In both cases, the best code found has $D^2_{\min} = 14\delta^2$; however, the search over codes having the structure shown in Fig. 5 took about five times longer than the search over codes having the structure shown in Fig. 3(a).

The best codes found for $\nu = 11$ and $\nu = 12$ are shown in Fig. 7. It was empirically observed that good codes can be found more rapidly by starting with a generator matrix with ones in entries not already designated (indicating a connection to a summer) and counting backwards, rather than starting with a generator matrix with all zeros and counting forwards. As an example, for $\nu = 12$, more than ten codes having $D^2_{\min} = 18\delta^2$ (coding gain of 6.3 dB) were found in less than 1/2 h on a Pyramid Model 98X machine; however, the search over all such codes was not completed in 5 h. It is empirically observed that if $D_{\min}$ for the best code with $2^\nu$ states is not due to the shortest possible error event (i.e., a single-error event), then doubling the number of states from $2^\nu$ to $2^{\nu+1}$ typically allows $D^2_{\min}$ to increase by $\delta^2$.

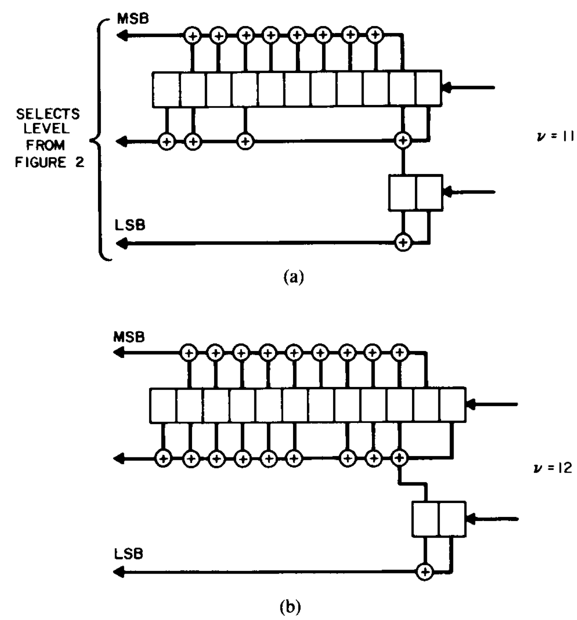


Fig. 7. (a) 8-AM code with $D^2_{\min} = 17\delta^2$ (coding gain = 6.1 dB). (b) 8-AM code with $D^2_{\min} = 18\delta^2$ (coding gain = 6.3 dB).

### B. Time-Varying Embedded Trellis Codes

The preceding search results suggest that a minimum of $2^{11}$ states is needed to obtain $D^2_{\min} = 17\delta^2$, whereas the best code with $2^7$ states and parallel transitions has $D^2_{\min} = 16\delta^2$. The techniques used to construct a two-state trellis code with 3 dB gain presented in [3] can be used, however, to construct a $2^9$-state 8-AM trellis code with $D^2_{\min} = 17\delta^2$.

Let $D'_{\min}$ denote the minimum free distance of a trellis code in which all single error events corresponding to parallel transitions are *excluded*. It follows that $D_{\min} \leq D'_{\min}$, and for the best known 8-AM trellis codes, $D'_{\min} = D_{\min}$ if $\nu \leq 7$. Consider two $2^{\nu-1}$-state trellis diagrams connected in parallel, as in Fig. 3(b), in which the signal points associated with transitions between parallel states contained in each of the two trellis diagrams are the same. The corresponding convolutional coder is shown in Fig. 8(b). This code is catastrophic, that is, assuming a transmitted sequence of all zeros, an error path may transit from trellis $T1$ to $T2$ and subsequently accumulate zero distance. This problem is avoided, however, by merging $T1$ and $T2$ every few symbols. In particular, after every $b$th symbol, only those transitions from $T2$ which are directed to $T1$ are allowed. Equivalently, a zero bit is input to the bottom shift register in Fig. 8(b) after every $b$th symbol. This reduces the transmission rate by one bit per $b$ symbols; however, $D_{\min}$ for the resultant $2^\nu$-state code is now the same as $D'_{\min}$ for the associated $2^{\nu-1}$-state rate 2/3 code with parallel transitions shown in Fig. 8(a) (assuming that this $D'_{\min}$ is less than or equal to the minimum distance incurred from any error event spanning two symbols).

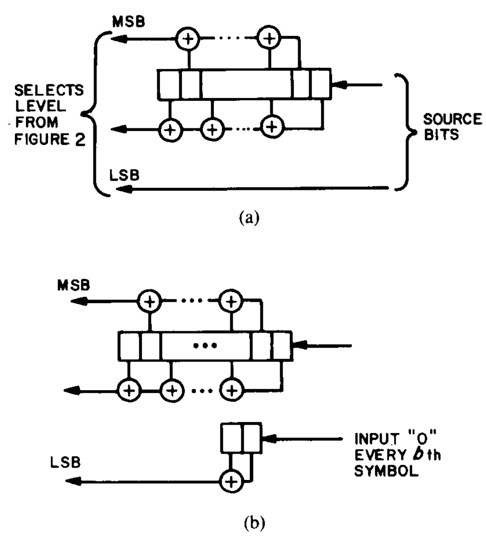


Fig. 8. (a) Rate 2/3 code with parallel transitions. (b) Rate 1/2 code embedded in a rate 2/3 time-varying code.

A computer search for $2^8$-state 8-AM trellis codes with

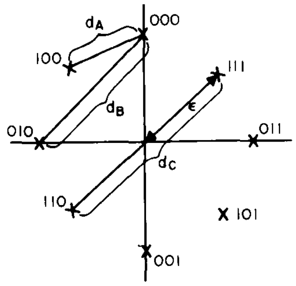


Fig. 9. 8-PSK signal constellation.

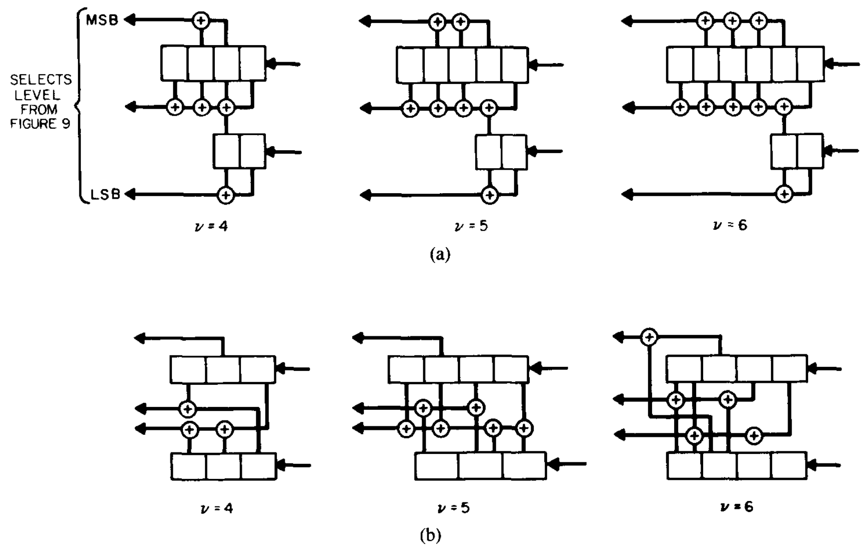


Fig. 10. (a) New 8-PSK codes. (b) 8-PSK Ungerboeck codes.

parallel transitions turned up at least one code for which $D'^2_{\min} = 17\delta^2 > D^2_{\min}$. The structure of such a code, which consists of a rate 1/2 code with an appended uncoded bit, is shown in Fig. 8(a). A $2^9$-state 8-AM trellis code with $D_{\min} = 17\delta^2$ can therefore be constructed by using the structure shown in Fig. 8(b), in which the embedded rate 1/2 codes (top shift registers) shown in Fig. 8(a) and (b) are the same. The two output bits of the top shift register select one of the four subconstellations at level $C$ in Fig. 2, and the bottom output bit selects one of the two constituent points. The corresponding trellis diagram consists of two identical rate 1/2 codes connected in parallel. Signal levels assigned to transitions contained in each trellis are negative ($-i\delta/2$, $i = 1, 3, 5, 7$), whereas signal levels assigned to transitions between the two trellis diagrams are positive. It is easily verified that the distance incurred in an error event spanning two symbols is $32\delta^2$. Consequently, (time-varying) 8-AM codes with $D^2_{\min}$ significantly greater than $17\delta^2$ can be found by searching over codes with parallel transitions and embedding the best codes in the structure shown in Fig. 8(b).

### C. 8-PSK Codes

Although the embedding technique has been illustrated for the AM signal constellation, it can easily be applied to any signal constellation. As another example, Fig. 10 shows rate 2/3 trellis codes combined with the 8-PSK signal constellation shown in Fig. 9. An 8-PSK trellis code with two parallel transitions between states must have $D_{\min} \leq d_C$ (see Fig. 9), which can be achieved for $\nu = 2$. Codes having the structures shown in Figs. 3(a) and 6 can easily be found which have the same $D_{\min}$ as the codes reported in [1] for $\nu > 2$. A few examples ($\nu = 4$, 5, and 6) are shown in Fig. 10(a). $D_{\min}$ for the $\nu = 6$ code is associated with the error event spanning two symbols ($D^2_{\min} = 6\epsilon^2$). The codes reported by Ungerboeck [1] are also shown.

## V. Conclusions

A new type of construction has been proposed for multilevel/phase trellis codes. Parallel transitions between states are essentially generalized to transitions between trellis diagrams in parallel. This method significantly reduces the size of the code space to be searched, and has produced codes having the same $D_{\min}$ as that for the previous best known codes with the same number of states, in addition to extending the range of reasonable search.

## References

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