On Waiting for Simultaneous Access to Two Resources: Deterministic Service Distribution

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Abstract—**Suppose that a *test customer* in an $M/D/1$ queueing system can get service only if he has access to the server and a separate event E has occurred. All other customers only require access to the server. The time until the event E occurs is assumed to be an exponentially distributed random variable. If the test customer reaches the server before E occurs, he must then return to the back of the queue. At any time, however, the test customer is allowed to give up his place in the queue and join the back of the queue. The test customer represents a computational task that depends upon the results of an associated task.**

The test customer's mean delay until service is derived assuming that he always maintains his position in the queue until he reaches the server. Conditions are given for which this "move-along" policy is optimal, i.e., minimizes the test customer's mean delay until service. A condition is also given for which the move-along policy is not optimal.

I. Introduction

The problem considered in this note is derived from a computing environment in which different but dependent computational tasks are to be scheduled for execution on multiple processors. Suppose $N > 1$ processors are used to process jobs submitted by multiple "parent" machines. A particular process W to be submitted by one parent machine is split into two smaller tasks W_1 and W_2 where W_2 depends on the result from running W_1, and each task must be run on two different processors P_1 and P_2. Assuming that jobs are generated from the parent machines according to a random process, queues may form at each processor. The time it takes to complete the process W is therefore random and depends on how many jobs are currently waiting for processors P_1 and P_2, and the time it takes to complete each job.

One scheme which minimizes the delay until W is completed requires that a special token be attached to W_2 so that if it reaches P_2 before W_1 is completed, it can allow other jobs access to P_2 while still maintaining its position at the head of the queue. In the more general situation where there are numerous processors and numerous jobs which require results from associated jobs, however, separate buffers are required for all jobs that reach the server but cannot be executed. In addition, a scheduler must keep track of which processes are related so that a job can get service as soon as the necessary input becomes available.

A simpler scheme is to submit both tasks W_1 and W_2 simultaneously to P_1 and P_2, respectively. If W_2 reaches P_2 before W_1 is completed, then W_2 is automatically placed at the back of P_2's queue. To analyze this scheme, a queueing model is constructed. Processors P_1 and P_2 can be regarded as servers for two different queueing systems, i.e., $G/G/1$ in the most general case. The task W_2 is a "test customer" who is waiting for a customer, W_1, to receive service from P_1. If the test customer reaches the server before W_1 is served, then he must join the back of the queue. At *any* time, however, the test customer is allowed to give up his current position and join the back of the queue. A specific problem studied here is to determine when, if ever, using this option reduces the mean delay until the test customer is served. Here we only consider the case where the arrival stream to server P_2 is a Poisson process, service times are deterministic ($M/D/1$ queueing system), and the time until W_1 is served is an exponentially distributed random variable, independent of the other queue. A general service distribution is considered in [2] and [3].

Another (possibly more appealing) interpretation of this problem [1] is that the test customer is waiting at a theater. He cannot get into the theater alone, however, because he does not have enough money. Nevertheless, he is counting on the arrival of a friend who will buy his ticket. If he reaches the cashier before his friend arrives, he must return to the back of the line. The test customer must decide whether or not to move to the back of the queue before he reaches the cashier in order to decrease his expected delay until admission.

Manuscript received April 8, 1986; revised January 21, 1987. Paper recommended by Past Associate Editor, A. Ephremides.
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IEEE Log Number 8714462.

II. The Main Results

Let t^* denote the *object time*, which is the first time the test customer reaches the server after his friend has arrived. At any given time $t < t^*$, the test customer is waiting in the queue. The state of the system at time t is therefore $[v(t), j(t), k(t)]$, where $v(t)$ is a real nonnegative number equal to the cumulative service times of customers, or *virtual work*, ahead of the test customer (including the customer currently being served), $j(t)$ is a nonnegative integer representing the number of customers in back of the test customer, and $k(t)$ is either 1 or 0, indicating, respectively, that the test customer's friend has, or has not, arrived. The state trajectory from time $t = 0$ to $t = T$ is defined as the continuum of states visited from time $t = 0$ to $t = T$, and is denoted as $s[0, T]$.

A *policy* P maps state trajectories to actions. For any policy P, the only actions allowed are either to stay in the current position, or jump to the back of the queue, i.e., move from state $[v(t), j(t), k(t)]$ to state $[v(t) + j(t), 0, k(t)]$. Suppose the state trajectory from $t = 0$ to $t = T$ is $s[0, T]$. The mean residual delay until the test customer is served starting from time T under policy P is denoted as $D(s[0, T]; P)$. The *move-along policy* (MAP) is defined as the policy whereby the test customer never leaves his position in the queue unless he has reached the head of the queue. The mean delay until the test customer is served under the MAP is denoted as $d_{v,j,k}$, where (v, j, k) is the current state. In this case the delay is independent of the current time t and the state trajectory prior to time t. Since $d_{v,j,1} = v$, we drop the k subscript, and write the mean delay until service, assuming the test customer's friend has not arrived, as $d_{v,j}$.

The first theorem gives a necessary and sufficient condition for which the MAP is optimal.

Theorem 1: Let $s[0, T]$ be any state trajectory which reaches state $(v, j, 0)$ at time T. Then $d_{v,j} = \inf_P D(s[0, T]; P)$ if and only if $d_{v,j} \le d_{v+j,0}$, for all positive v and j.

This theorem holds for all previously defined policies P. If $d_{v,j} > d_{v+j,0}$ for particular v and j, then moving to state $(v + j, 0, 0)$, rather than staying in state $(v, j, 0)$, decreases the mean delay relative to the move-along mean delay. In this case, policy iteration [4] can be used to obtain the best policy.

A discrete version of Theorem 1 in which the parameter v only takes on discrete values, and policy decisions are made at each successive time step, essentially follows from Theorem 1.1 in [4, ch. 3], and the fact that the object time is finite with probability one. Theorem 1 is obtained by letting the time steps decrease to zero, and arguing that for any allowable policy P, there exists a sequence of discrete-time policies P_i such that $\lim_{i\to\infty} D(s[0, T]; P_i) = D(s[0, T]; P)$. Details are omitted.

To decide whether or not the MAP is optimal the mean delay $d_{v,j}$ is explicitly computed. Let λ denote the arrival rate to the $M/D/1$ queue, α denote the rate at which the test customer's friend arrives, and assume that the service rate μ is normalized to one. It is shown in Appendix A that $d_{v,j}$ satisfies the recursion

$$d_{v,j} = v + e^{-(\lambda+\alpha)v} \sum_{k=0}^{\infty} \frac{(\lambda v)^k}{k!} d_{j+k,0}. \tag{1}$$

For $j = 0$, this expression reduces to

$$d_{v,0} = v + e^{-(\lambda+\alpha)v} d_{0,0} + e^{-(\lambda+\alpha)v} \sum_{k=1}^{\infty} \frac{(\lambda v)^k}{k!} d_{k,0}. \tag{2}$$

The following boundary condition holds for $d_{0,0}$:

$$d_{0,0} = \frac{1}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha} d_{1,0}. \tag{3}$$

0018-9286/87/1100-1022$01.00

The first term on the right-hand side is the average time until either the next arrival to the queue, or event E occurs, and the term multiplying $d_{1,0}$ is the probability that event E occurs before the next arrival. The solution to (2), derived in Appendix A, can be written as

$$d_{v,0}=e^{-vx_\infty}d_{0,0}+\sum_{k=0}^{\infty}\lambda^k v\exp\left[-vx_k-\sum_{m=0}^{k-1}x_m\right] \tag{4}$$

where

$$x_{k+1}=\alpha+\lambda(1-e^{-x_k}),\quad x_0=0, \tag{5}$$

and

$$x_\infty=\lim_{k\to\infty}x_k, \tag{6a}$$

that is,

$$x_\infty=\alpha+\lambda(1-e^{-x_\infty}),\quad x_\infty>0. \tag{6b}$$

Combining (3) and (4) gives

$$d_{0,0}=\frac{1}{x_\infty}\left[1+\sum_{k=0}^{\infty}\lambda^{k+1}\exp\left(-\sum_{m=0}^{k}x_m\right)\right]. \tag{7}$$

Once $d_{0,0}$ is computed from (5)–(7), $d_{v,0}$, $v>0$, can be computed from (4)–(6), where the infinite sums are truncated.

Plots of $d_{v,0}$ as a function of v for two sets of λ and α are shown in Figs. 1 and 2. Also shown in each case is the curve $d_v = v + e^{-\alpha v}/\alpha$, which is the minimum possible mean delay obtained by allowing the test customer to wait at the head of the queue. The "hump" in Fig. 2 is due to the relatively large value of λ, and suggests Theorem 3, which follows.

Computation of $d_{v,j}$ for $j>0$ merely requires substituting the expression for $d_{j+k,0}$, given by (4), into (1). Although somewhat messy, this substitution is straightforward, and the details are omitted. The result is

$$d_{v,j}=v+e^{-x_\infty(v+j)}d_0+\sum_{k=0}^{\infty}\left[\lambda^k(j+\lambda ve^{-x_k})\exp\left(-jx_k-\sum_{m=0}^{k-1}x_m-vx_{k+1}\right)\right]. \tag{8}$$

Given Theorem 1 and the expression for mean delay (8), it is now possible to determine a condition on λ and α which guarantees that the MAP is optimal. Proofs of Theorems 2, 3, and 4 are given in Appendix B.

Theorem 2: If $\lambda \le \alpha/(1-e^{-\alpha})$, then $d_{v,j} \le d_{v+j,0}$ for all positive v and j.

Observe that $\alpha/(1-e^{-\alpha}) > 1$ for all $\alpha > 0$ so that if the queue is stable, then the test customer cannot decrease his mean delay until service by moving to the back of the queue. The next theorem implies that if λ is large enough, however, then the MAP is not optimal.

Theorem 3: Given any α, there exists a threshold $\lambda_0(\alpha)$, such that if $\lambda > \lambda_0(\alpha)$, then $d_{v+j,0} < d_{v,j}$ for some v and j.

Theorem 3 is illustrated by the following plausibility argument. Referring to Fig. 2, suppose that $\lambda = 100$, $\alpha = 0.1$, and the initial state is (10, 0, 0), i.e., 10 customers ahead of the test customer. If the test customer adopts the MAP, the probability that he will reach the server before his friend arrives is $e^{-1} \approx 0.37$, in which case about 1000 new customers appear, so that his expected delay is quite large (i.e., approximately 380 for $\lambda = 100$). If, however, the test customer waits only for the total number of customers in the queue to increase, say, to 60 before moving to the back of the queue, his friend will with high probability arrive before he reaches the server, so that his expected delay will decrease significantly (i.e., to approximately 75 from Fig. 2).

The previous two theorems suggest the following unproven conjecture.

Conjecture: There exists a λ^*, which depends on α, such that the MAP is optimal if and only if $\lambda \le \lambda^*(\alpha)$.

Define $\bar\lambda(\alpha)$ as the infimum of all threshholds λ_0 referred to in Theorem 3, and let

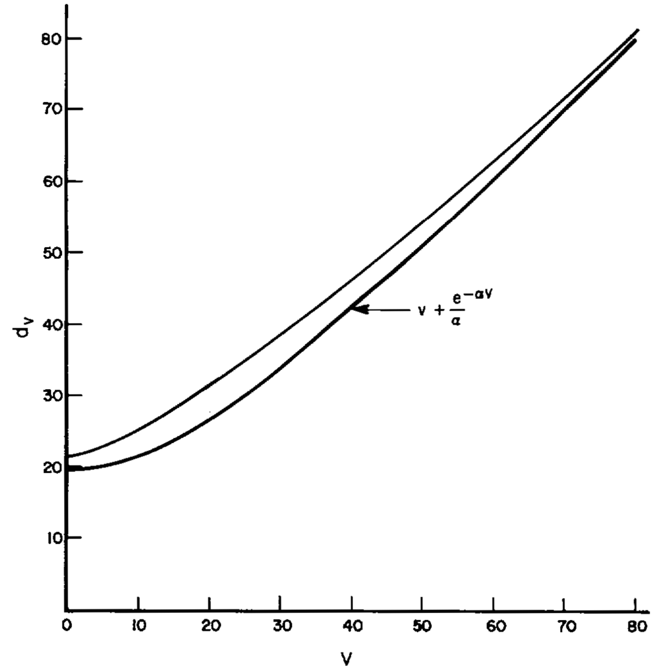

Fig. 1. Plot of d_v versus v with $\lambda = 1$ and $\alpha = 0.05$.

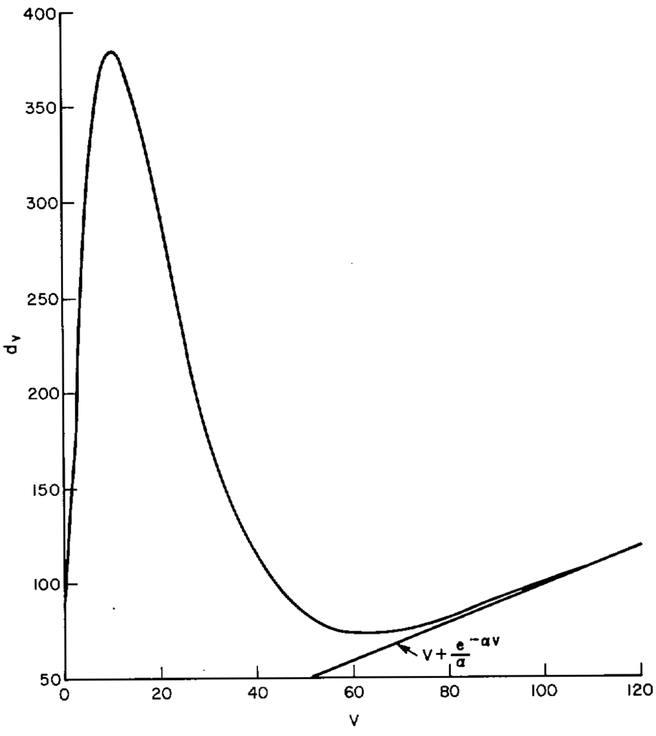

Fig. 2. Plot of d_v versus v with $\lambda = 100$ and $\alpha = 0.1$.

$$g(\alpha) \equiv \frac{\alpha}{1-e^{-\alpha}}. \tag{9}$$

Theorem 2 implies that $\bar\lambda(\alpha) \ge g(\alpha)$. The next theorem states that this bound is tight for large α, and for α close to zero.

Theorem 4:

i) $\lim_{\alpha\to 0}\bar\lambda(\alpha)=1$

ii) $\lim_{\alpha\to\infty}[\bar\lambda(\alpha)-g(\alpha)]=0.$

We remark that the lower bound $\bar{\lambda}(\alpha) \geq g(\alpha)$ is fairly tight for moderately large α, i.e., the analysis in Appendix B implies that $\bar{\lambda}(4) - g(4) < 0.3$.

III. GENERALIZATIONS

The problem considered here has many interesting variations and generalizations. A general, rather than deterministic, service distribution is considered in [2] and [3]. Different distributions for the arrival time of the test customer's friend could also be considered. One example of this is a nested problem in which the test customer's friend is waiting in another queue (i.e., at a bank) and cannot be served until a third person arrives (to grant approval of a cash withdrawal). This type of nesting can be increased to any finite or infinite level. A similar variation assumes the test customer's friend is in another Markov chain and that the test customer cannot be served before his friend reaches a specific state. A generalization which is perhaps more closely tied to a computer environment is one in which there are several test customers in the queue. What is the optimal policy for each test customer?

APPENDIX A

DERIVATION OF (1)–(6)

To derive (1) let $d_{v,j|T}$ denote the mean delay from state $(v, j, 0)$ given that the friend arrives at time T. For $v > 0$

$$d_{v,j|T} = \begin{cases} v & T < v \\ v + \sum_{k=0}^{\infty} d_{j+k,0|T-v} \frac{(\lambda v)^k}{k!} & T > v. \end{cases} \tag{A.1}$$

Since the arrival time of the test customer's friend is exponentially distributed with parameter α,

$$\begin{aligned} d_{v,j} &= \int_0^{\infty} \alpha e^{-\alpha T} d_{v,j|T}\, dT \\ &= v + \sum_{k=0}^{\infty} \frac{(\lambda v)^k}{k!} e^{-\lambda v} \int_v^{\infty} \alpha e^{-\alpha T} d_{j+k,0|T-v}\, dT \\ &= v + e^{-(\lambda+\alpha)v} \sum_{k=0}^{\infty} \frac{(\lambda v)^k}{k!} d_{j+k,0}. \end{aligned} \tag{A.2}$$

A solution to (2) and (3) can be obtained by defining the sequence $d_{v,0}^{(i)}$, $i = 0, 1, \cdots$, as

$$d_{v,0}^{(i+1)} = v + e^{-v(\lambda+\alpha)} \left(d_{0,0} + \sum_{k=1}^{\infty} \frac{(\lambda v)^k}{k!} d_{k,0}^{(i)} \right) \tag{A.3a}$$

$$d_{v,0}^{(0)} = 0. \tag{A.3b}$$

It is easily shown that the solution to (2) and (3) is unique, and that $d_{v,0}^{(i)}$ monotonically increases with i and converges to $d_{v,0}$. Iterating (A.3) a few times gives:

$$d_{v,0}^{(1)} = v + e^{-v(\lambda+\alpha)} d_{0,0} \tag{A.4a}$$

$$d_{v,0}^{(2)} = v + \lambda v e^{-v\alpha} + e^{-v(\lambda+\alpha)} e^{\lambda v e^{-(\lambda+\alpha)}} d_{0,0} \tag{A.4b}$$

$$d_{v,0}^{(3)} = v + \lambda v e^{-v\alpha} + \lambda^2 v e^{-v\lambda(1-e^{-\alpha}) - \alpha(v+1)} + e^{-v(\lambda+\alpha)} e^{\lambda v e^{-(\lambda+\alpha)+\lambda e^{-(\lambda+\alpha)}}} d_{0,0}. \tag{A.4c}$$

Examining the sequence $d_{v,0}^{(i)}$, $i = 1, 2, \cdots$ yields the series expansion (4)–(6).

APPENDIX B

PROOFS OF THEOREMS 2–4

Theorems 2–4 rely on the following lemma.

Lemma: The sequence x_k increases monotonically with k. Also,

$$\alpha \leq x_k \leq \lambda + \alpha, \qquad k \geq 1. \tag{B.1}$$

Proof: From (5),

$$x_0 = 0 < x_1 = \alpha.$$

Also,

$$x_{k+1} - x_k = \lambda(e^{-x_{k-1}} - e^{-x_k}). \tag{B.2}$$

Assuming $x_{k-1} < x_k$, then $e^{-x_{k-1}} > e^{-x_k}$, and $x_k < x_{k+1}$. Inequality (B.1) follows directly from (5) since $e^{-x_k} < 1$. □

Proof of Theorem 2: From (4) and (8),

$$\begin{aligned} d_{v+j,0} - d_{v,j} &= j + \sum_{k=0}^{\infty} \left\{ \lambda^{k+1}(v+j) \exp\left[-\sum_{m=0}^{k} x_m - (v+j)x_{k+1} \right] \right. \\ &\quad \left. - \lambda^k (j + \lambda v e^{-x_k}) \exp\left[-jx_k - vx_{k+1} - \sum_{m=0}^{k-1} x_m \right] \right\} \\ &= j + \sum_{k=0}^{\infty} \lambda^k \exp\left[-vx_{k+1} - \sum_{m=0}^{k-1} x_m \right] \\ &\quad \cdot [\lambda v e^{-x_k}(e^{-jx_{k+1}} - e^{-jx_k}) + j(\lambda e^{-x_k - jx_{k+1}} - e^{-jx_k})] \\ &= j + \sum_{k=0}^{\infty} \lambda^k \exp\left(-\sum_{m=0}^{k} x_m \right) [je^{-vx_{k+1}}(\lambda e^{-jx_{k+1}} - e^{-(j-1)x_k}) \\ &\quad - \lambda v e^{-vx_{k+1}}(e^{-jx_k} - e^{-jx_{k+1}})] \\ &= \sum_{k=0}^{\infty} \lambda^k \exp\left(-\sum_{m=0}^{k} x_m \right) [je^{-(j-1)x_k}(e^{-vx_k} - e^{-vx_{k+1}}) \\ &\quad - \lambda v e^{-vx_{k+1}}(e^{-jx_k} - e^{-jx_{k+1}})]. \end{aligned} \tag{B.3}$$

Let

$$f(x_k, x_{k+1}) \equiv je^{-(j-1)x_k}(e^{-vx_k} - e^{-vx_{k+1}}) - \lambda v e^{-vx_{k+1}}(e^{-jx_k} - e^{-jx_{k+1}}). \tag{B.4}$$

The sum (B.3) will be nonnegative if $f(x_k, x_{k+1}) \geq 0$ for all $k \geq 0$. Observe that

$$f(x_k, x_k) = 0$$

and

$$\frac{\partial f}{\partial x_k} = -j(j-1)e^{-(j-1)x_k}(e^{-vx_k} - e^{-vx_{k+1}}) - jve^{-jx_k}(e^{-(v-1)x_k} - \lambda e^{-vx_{k+1}}) \tag{B.5}$$

which from the lemma is negative if

$$\lambda \leq e^{v(x_{k+1} - x_k) + x_k}. \tag{B.6}$$

Consequently, if λ is a constant less than or equal to e^{x_k}, then $f(x_k, x_{k+1})$ is nonnegative. The lemma therefore implies that $f(x_k, x_{k+1}) \geq 0$ for all $k \geq 1$ if $\lambda \leq e^{\alpha}$. For $k = 0$,

$$f(x_0, x_1) = f(0, \alpha) = j(1 - e^{-v\alpha}) - \lambda v e^{-v\alpha}(1 - e^{-j\alpha}), \tag{B.7}$$

which is nonnegative for all j and v if

$$\lambda \leq \min_{j \geq 1, v > 0} \frac{j}{1 - e^{-j\alpha}} \frac{1 - e^{-v\alpha}}{ve^{-v\alpha}}. \tag{B.8}$$

It is easily verified that the right-hand side is minimized by setting $j = 1$ and letting v approach zero, so that $f(x_k, x_{k+1}) \geq 0$ for all $k > 0$ if

$$\lambda \leq \frac{\alpha}{1-e^{-\alpha}} < e^{\alpha} \tag{B.9}$$

for $\alpha > 0$. □

Before proving Theorem 3, we remark that for $v = j = 1$, (B.3) becomes

$$d_{2,0} - d_{1,1} = \sum_{k=0}^{\infty} \lambda^k \exp\left(-\sum_{m=0}^{k} x_m\right)(e^{-x_k} - e^{-x_{k+1}})(1-\lambda e^{-x_{k+1}}). \tag{B.10}$$

If $\lambda > e^{\alpha}$, then the first term in the sum will be negative. However, it is *not* true that all of the remaining terms become negative for large enough λ. In particular, $\lambda e^{-x_2} = \lambda \exp(-\lambda(1-e^{-\alpha}) - \alpha) < 1$ for large enough λ, and $\lambda e^{-x_2} > \lambda e^{-x_3} > \cdots > \lambda e^{-x_\infty} > 0$. It therefore may be true that the sum (B.10) is positive for *all* λ.

Proof of Theorem 3: From (5), (B.3), and the preceding lemma,

$$\begin{aligned} d_{v+1,0} - d_{v,1} < \delta(v) \equiv{}& (1-e^{-v\alpha}) - \lambda v e^{-v\alpha}(1-e^{-\alpha}) \\ &+ \lambda e^{-\alpha}(e^{-\alpha v} - e^{-x_2 v} - \lambda v e^{-v x_2}(e^{-\alpha} - e^{-x_2})) \\ &+ \lambda^2 e^{-\alpha - x_2} \sum_{k=2}^{\infty} (\lambda e^{-x_2})^{k-2}(e^{-v x_2} - e^{-v x_\infty}) \\ ={}& (1-e^{-v\alpha}) - \lambda v e^{-v\alpha}(1-e^{-\alpha}) \\ &+ \lambda e^{-\alpha}(e^{-\alpha v} - e^{-x_2 v} - \lambda v e^{-v x_2}(e^{-\alpha} - e^{-x_2})) \\ &+ (e^{-v x_2} - e^{-v x_\infty}) \frac{\lambda^2 e^{-\alpha - x_2}}{1-\lambda e^{-x_2}} \end{aligned} \tag{B.11}$$

assuming λ is large enough so that $\lambda e^{-x_2} < 1$. For fixed v and α, as λ increases, the terms containing x_2 approach zero, so that

$$\delta(v) \to 1 - e^{-v\alpha} - \lambda[v e^{-\alpha v}(1-e^{-\alpha}) - e^{-\alpha(v+1)}] \tag{B.12}$$

which is negative for large enough λ provided that

$$v > \frac{1}{e^{\alpha} - 1}. \tag{B.13}$$

Consequently, for any fixed v which satisfies (B.13), there exists a λ_0 such that $\lambda > \lambda_0$ implies $d_{v+1,0} - d_{v,1} < \delta(v) < 0$. ◻

Proof of Theorem 4: Since $\lim_{\alpha \to 0} g(\alpha) = 1$, it follows that $\lim_{\alpha \to 0} \bar{\lambda}(\alpha) \geq 1$. Also, $\lim_{\alpha \to 0} \lambda e^{-x_k} = \lambda$ for all $k \geq 0$, so that if $\lambda > 1$, then each term in the sum (B.10) becomes negative for small enough α. Therefore, $\lim_{\alpha \to 0} \bar{\lambda}(\alpha) \leq 1$, which proves i).

To prove ii), we first examine the conditions for which $\delta(v)$, defined in (B.11), is negative for small v. Observe that $\delta(0) = 0$. Also, from (B.11) after some manipulation,

$$\begin{aligned} \left.\frac{d\delta(v)}{dv}\right|_{v=0} &= \alpha - \lambda(1-e^{-\alpha}) + \lambda^2 e^{-\alpha}(1 - 2e^{-\alpha} + e^{-x_2}) + (x_\infty - x_2)\frac{\lambda^2 e^{-\alpha - x_2}}{1-\lambda e^{-x_2}} \\ &< \alpha - \lambda(1-e^{-\alpha}) + [\lambda(1-e^{-\alpha})]^2 e^{-\alpha} + \epsilon(\alpha) \end{aligned} \tag{B.14}$$

where

$$\epsilon(\alpha) = (x_\infty - x_2)\frac{\lambda^2 e^{-\alpha - x_2}}{1-\lambda e^{-x_2}} \tag{B.15}$$

and the inequality follows from the assumption $\lambda > g(\alpha)$, defined by (9), so that $e^{-x_2} = e^{-\lambda(1-e^{-\alpha})-\alpha} < e^{-2\alpha}$. It is easily verified that $\lambda^2 e^{-\alpha - x_2}$ and λe^{-x_2} are decreasing functions of λ for $\lambda > 2/(1 - e^{-\alpha})$. Consequently, if $\alpha > 2$, λ can be replaced by $g(\alpha)$ in (B.15) to give

$$\epsilon(\alpha) < x_\infty \frac{\alpha^2 e^{-3\alpha}}{1 - 2e^{-\alpha} + e^{-3\alpha}}. \tag{B.16}$$

Since $\lim_{\alpha \to \infty} x_\infty = 0$, $\epsilon(\alpha)$ is bounded by an exponentially decaying function for large α.

Setting the right side of (B.14) less than zero, and solving the quadratic inequality gives

$$\begin{aligned} \underline{\gamma}(\alpha) &\equiv \frac{2\alpha}{(1-e^{-\alpha})\{1 + \sqrt{1 - 4[\alpha + \epsilon(\alpha)]e^{-\alpha}}\}} \\ &< \frac{2\alpha}{(1-e^{-\alpha})\{1 - \sqrt{1 - 4[\alpha + \epsilon(\alpha)]e^{-\alpha}}\}} \equiv \bar{\gamma}(\alpha) \end{aligned} \tag{B.17}$$

assuming that

$$4[\alpha + \epsilon(\alpha)]e^{-\alpha} \leq 1 \tag{B.18}$$

which is true for $\alpha \geq 2.2$. Now $\lim_{\alpha \to \infty} [\underline{\gamma}(\alpha) - g(\alpha)] = 0$, so that for large enough α, if $\lambda > g(\alpha)$, then $d_{v+1,0} - d_{v,1} < \delta(v) < 0$ for v close to zero. The right-hand inequality indicates, however, that if $\lambda > \bar{\gamma}(\alpha)$, then $\delta(v)$ is positive for v close to zero. It therefore remains to be shown that $\bar{\lambda}(\alpha) \leq \bar{\gamma}(\alpha)$ for large α.

Substituting $v = 1/\alpha$ in (B.11) gives

$$\delta(1/\alpha) < 1 - e^{-1} + \lambda e^{-1}\left(-\frac{1}{\alpha}(1-e^{-\alpha}) + e^{-\alpha}\right) + \epsilon'(\alpha) \tag{B.19}$$

where

$$\epsilon'(\alpha) = (e^{-x_2/\alpha} - e^{-x_\infty/\alpha})\frac{\lambda^2 e^{-\alpha - x_2}}{1-\lambda e^{-x_2}} < \epsilon(\alpha), \tag{B.20}$$

defined by (B.15). From (B.19) it follows that $\delta(1/\alpha) < 0$ if

$$\lambda > \frac{\alpha[(e-1) + e\epsilon'(\alpha)]}{1 - e^{-\alpha}(1+\alpha)} \tag{B.21}$$

which is smaller than $\bar{\gamma}(\alpha)$, defined by (B.17), for large enough α (i.e., $\alpha \geq 2.5$), so that $\bar{\lambda}(\alpha) \leq \underline{\gamma}(\alpha)$, which proves ii). ◻

ACKNOWLEDGMENT

The author especially thanks B. Gopinath for providing the problem, T. Ott for providing a first proof of Theorem 2, and D. Cohen, S. Y. R. Li, and E. Arthurs for helpful conversations.

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State-Space Discrete Systems: 2-D Eigenvalues

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Abstract—**A correction is given to a proof of the fact, presented in a paper by Roesser [1], that every partitioned matrix A not only satisfies the two-dimensional characteristic equation, but it must also satisfy an**

Manuscript received June 30, 1986.
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IEEE Log Number 8714289.

0018-9286/87/1100-1025$01.00