On the Nature of Revenue-Sharing Contracts to Incentivize Spectrum-Sharing

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***Abstract*—In a limited form cellular providers have long shared spectrum in the form of roaming agreements. The primary motivation for this has been to extend the coverage of a wireless carrier's network into regions with no infrastructure. As devices and infrastructure become more agile, such sharing could be done on a much faster time-scale and have advantages even when two providers both have coverage in a given area, e.g., by enabling one provider to acquire "overflow" capacity from another provider during periods of high demand. This may provide carriers with an attractive means to better meet their rapidly increasing bandwidth demands. On the other hand, the presence of such a sharing agreement could encourage providers to under-invest in their networks, resulting in poorer performance. We adapt the newsvendor model from the operations management literature to model such a situation and to gain insight into these trade-offs. In particular, we analyze the structure of revenue-sharing contracts that incentivize both capacity sharing and increased access for end-users.**

I. Introduction

Capacity sharing in the form of roaming agreements [1]–[3] have long been a fixture of cellular service. The main historical impetus for this type of capacity sharing was coverage extension. By entering into a roaming agreement, the customers of a wireless carrier could receive service in regions where that carrier had no infrastructure. This makes a carrier's service more attractive to customers who "roam" outside of the carrier's coverage, without requiring the carrier to invest in coverage expansion. Here, we focus on a different form of capacity sharing, namely acquiring "overflow" capacity from another carrier during periods of high demand. Such sharing may provide carriers with an attractive means to better meet their rapidly increasing bandwidth demands and would likely result in more efficient spectrum utilization. On the other hand, as with traditional roaming agreements, the presence of such sharing agreements can effect a carrier's incentives to invest, e.g., knowing that one could send excess traffic to another carrier's network, could lead a carrier to under-invest in their own network, resulting in poorer performance. In this paper we seek to understand such trade-offs. To accomplish this, we consider a stylized model of such a situation based on the classic *newsvendor model* from operations management [4]–[6].

In the newsvendor model, a single firm seeks to determine how much inventory to stock in the face of uncertain demand. There is a cost to procuring each unit of inventory, which can in turn be sold for a fixed price provided that there is demand for it. Any inventory that exceeds the realized demand cannot be sold. Here, we consider two non-cooperative wireless carriers, who are determining how much capacity to invest in, also in the face of uncertain demand. Without any capacity sharing, this can be modeled as the standard newsvendor model for each carrier. However, with sharing, the carriers' investment decisions become coupled since the revenue one carrier earns from sharing capacity with another depends on the other's capacity investment. We model the interaction due to sharing as a game in which each carrier invests so as to maximize its expected profit under a given sharing rule. Two different sharing models are considered: one in which a carrier first serves its own customers before allocating any excess capacity to the other provider's customers and one in which carriers do not discriminate between their own traffic and the other carrier's. For each model we given conditions under which the resulting games have a unique pure strategy Nash equilibrium and are also *individually rational*, i.e., each carrier's expected profit is higher than it would be without sharing. In particular, when both providers receive the same revenue per unit demand a unique Nash equilibrium always exists for the first sharing model, as well as for the second model when one provider gets all of the revenue from sharing.

Our analysis allows for the demands seen by the two providers to be correlated and only requires that a joint density exists. We also consider the extreme cases where the demands of the two providers are either co-monotone or counter-monotone (in which case no joint density exists). We also show that the Nash equilibrium can be calculated via iterative best responses. When the revenue per unit demand is equal and sharing contract satisfies a type of symmetry, we further show that the Nash equilibrium is given by a solution to a convex optimization problem. Finally, based on our analysis we present numerical results comparing the amount of investment with and without sharing agreements. These results show that the amount of investment with sharing an be either greater than or less than the amount with sharing. The key factor in determining this is how the revenue from sharing is split between the providers. When more of this revenue goes to the spectrum owner, the owner has greater incentive to invest. The dependence in the demands of the two providers also plays a role; counter-monotone demands providing a greater incentive to invest. However, even if the demands are co-monotone, there can still be an incentive to increase investment provided the spectrum owner gets a large enough share of the revenue.

In terms of related work, there has been a line of literature studying how carriers set prices in roaming agreements (see e.g. [7]–[13]). Much of this work focuses on a known demand and seeks to understand if firms pricing decision lead to collusive behavior [7], [10]–[17]. An extension of this line of work that also considers investment can be found in [18]. Here we assume that prices for roaming are given exogenously and focus instead on the carriers investment decisions in the face of uncertain demand. There has also been increasing interest in sharing "raw spectrum" between different providers (e.g. [19]). This differs from the capacity sharing model considered here in that if a carrier receives raw spectrum from another carrier, it would still have to use its own infrastructure to utilize this spectrum. Finally, we note that the type of capacity sharing model considered here may be applicable to other settings as well, such as sharing between providers of renewable energy in real-time electricity markets [20] where electricity/energy can be re-sold in real-time.

The paper is organized as follows. In Section II we describe the newsvendor model and its application to our problem. Sections III and IV describe two models of sharing and our results. We present in Section V with some numerical examples to illustrate our results and some subtleties therein, and conclude in Section VI.

II. Non-Sharing Model

We consider a situation with two carriers, 1 and 2, who are faced with unknown demands in each of their markets. They each procure capacity, e.g., by investing in new infrastructure or buying spectrum on a secondary market. Each carrier i ($i = 1, 2$) receives a revenue of p_i per unit demand that it can serve and pays a cost per unit capacity of c_i. The demands are denoted by a non-negative stochastic quantity D_i with probability density function $f_i(\cdot)$ and cumulative distribution function $F_i(\cdot)$ (complementary cumulative distribution function being $F_i^C(\cdot)$).[1] The problem facing each carrier i is to determine an amount of capacity q_i to buy, knowing only the distribution of D_i but not the actual value. Initially, to establish a base-line case, we consider a scenario in which there is no sharing between the carriers, and the demands are independent. Each carrier's problem can then be viewed as an instance of the newsvendor model [4]–[6].

With no sharing, for a given choice of q_i, the expected profit of carrier i is given by

$$\begin{aligned} E[\tilde{\pi}_i^{NS}] &= p_i E[\min(q_i, D_i)] - c_i q_i \\ &= p_i q_i F_i^C(q_i) + p_i \int_0^{q_i} x f_i(x) dx - c_i q_i \end{aligned}$$

[1]Of course, depending on the time-scale at which procurement occurs, the actual demand may be a time-varying quantity, in which case one should interpret D_i in an average sense.

where the superscript NS denotes non-sharing. Since $\min(q_i, x) =: q_i \wedge x$ is a concave function of q_i for any $x \in (-\infty, \infty)$, it follows that $p_i E[\min(q_i, D_i)]$ is also a concave function of q_i. If, in addition, a density exists for $F_i(\cdot)$, then by taking derivatives, it is easy to see that the expected profit is a strictly concave function of q_i and the optimal amount of capacity purchased is given by

$$q_i^{*,NS} = F_i^{-1}\left(1 - \frac{c_i}{p_i}\right)$$

where $F_i^{-1}(\cdot)$ are inverse functions given by

$$F_i^{-1}(y) = \inf\{x : F_i(x) = y\}.$$

Denote the optimal profit as $E[\tilde{\pi}_i^{*,NS}]$ for future reference. Subsequently, we will view this as the outside option available to the carriers, i.e., this is the profit they would receive if they did not participate in a sharing agreement. Note that the analysis carries over unchanged even when the demands are dependent random variables with the marginal distributions having densities; unless specified otherwise, this will be the standing assumption in the rest of this paper.

III. Sharing Model A

We now allow the carriers to share their excess spectrum, i.e., if one of them has excess demand and the other excess spectrum, then the excess spectrum can be applied to the excess demand with the spectrum holder charging for usage of her spectrum. For explaining the capacity/spectrum sharing scheme, let us assume the instantiation of the demands is such that $q_1 < D_1$ and $q_2 \geq D_2$ so that carrier 2 has excess spectrum that can be applied to the excess demand that carrier 1 sees. Assume that carrier 2 charges a price c_r per unit of capacity. Then she makes an additional profit of $c_r \min(D_1 - q_1, q_2 - D_2)$ while carrier 1 makes an additional profit of $(p_1 - c_r)\min(D_1 - q_1, q_2 - D_2)$. Here we assume that the carriers can distinguish the different types of traffic and serve all the native demand before serving the traffic of the other carrier. We will assume that $c_r \leq p_1$; otherwise, the expected profit of carrier 1 would decrease, giving her no incentive to share this traffic. With such an assumption in place, the maximum amount of money made from the transaction is $p_1 \min(D_1 - q_1, q_2 - D_2)$, and this has to be shared between the carriers in the some fashion. We will assume in the current scenario that carrier 1 gets an $\alpha \in [0, 1]$ fraction of this money with the rest going to carrier 2. In the opposite scenario where carrier 1 has the excess spectrum and carrier 2 the excess demand, we will assume that carrier 1 gets a $1 - \beta$ fraction of the money, which equals $p_2 \min(D_2 - q_2, q_1 - D_1)$, with carrier 2 retaining the remainder of the money. An α (or β) less than a half implies that the owner of the spectrum keeps more of the money to be made from the sharing contract, and the *vice-versa* for the case of α greater than a half. This will be an important distinction in the analysis later on.

Unlike in the non-sharing case, we now have a game that the two carriers play. They will try to choose their purchases of spectrum q_i using the knowledge of the distributions of the demands of both carriers and the sharing mechanism in place. The timing of the game is as follows: first the prices are given and sharing contract is fixed, i.e., the p_is, c_is, α and β parameters are given; next the carriers simultaneously purchase capacity/spectrum with the knowledge of the contract and the distribution of the demands; and finally, the demands are revealed resulting in each player receiving an *ex post* pay-off equal to their resulting profit. In such a setting we will explore the existence of pure strategy Nash equilibria [21]; since we are analyzing purchase of spectrum/capacity, mixed equilibria are hard to justify in a practical setting. As mentioned before, if one includes the bigger question of whether carriers will be incentivized to share spectrum, then one must include the constraint that their profits are above what they can obtain in the no-sharing scenario, i.e., sharing must be individually rational. This will be discussed further in the analysis of the resulting equilibria. We first present results assuming that the demands have a joint probability function with a density. We establish some notation first. Let $F(x, y)$ be the joint cumulative distribution function, i.e., $P(X \leq x, Y \leq y)$, with joint density $f(x, y)$. Then the corresponding marginals are given by

$$F_1(x) = F(x, \infty), \quad F_2(x) = F(\infty, x)$$
$$f_1(x) = \int_0^\infty f(x, y)dy, \quad f_2(y) = \int_0^\infty f(x, y)dx.$$

Towards the end this section we will also consider specific joint distributions where marginal densities exist but a joint density does not exist.

Since the carriers do not know the realization of the demand when choosing their investments, we assume that they each attempt to maximize their expected profits (i.e., they are risk neutral). The expected profit of carrier 1 when she purchases q_1 and carrier 2 purchases q_2 is given by

$$\begin{aligned}E[\pi_1^A] &= p_1 q_1 F_1^C(q_1) + p_1 \int_0^{q_1} x f_1(x)dx - c_1 q_1 \\ &+ \alpha p_1 \int_{q_1}^\infty \int_0^{q_2} \min(y - q_1, q_2 - y) f(x, y)dydx \\ &+ (1-\beta)p_2 \int_0^{q_1} \int_{q_2}^\infty \min(q_1 - x, y - q_2) f(x, y)dydx\end{aligned}$$

where superscript A stands for model A of sharing. Here, the first two terms on the right-hand side are the revenue a carrier makes from its own customers. The third term is the cost of investment. The fourth and fifth terms are the revenue from sharing when carrier 1 and carrier 2 have excess demands, respectively. The expected profit for carrier 2 can be written in a similar manner. The analysis proceeds by determining the best-response of carrier 1 when carrier 2 purchases spectrum q_2, and *vice-versa*. We will rule out the cases $(\alpha = 1, \beta = 0)$ and $(\alpha = 0, \beta = 1)$ where one carrier gets all the profit from sharing. In these cases the strategy is obvious: one carrier purchases as per the no-sharing scenario and the other carrier purchases spectrum taking into account this action. The main result is then the following.

Theorem 1: The spectrum game outlined above has a unique pure Nash equilibrium $(q_1^{*,A}, q_2^{*,A})$ if $p_1 \geq (1-\beta)p_2$ and $p_2 \geq (1-\alpha)p_1$. In addition, the equilibrium can be obtained by iterating the best-response correspondences.

Note that p_1 is the revenue per customer carrier 1 receives from its own customers, while $(1-\beta)p_2$ is the revenue per customer it receives for each customer of carrier 2's that is served. Thus the conditions in this theorem are simply stating that each provider does not receive more revenue from serving a customer of the other provider. Also note that this is always satisfied when both providers have the same revenue per unit demand $(p_1 = p_2)$.

Proof: The proof is carried out in two steps. The first step shows that the profit maximization problem of each carrier is a convex optimization problem with a unique solution. This then establishes the continuity of each carrier's best response in the choice made by the other carrier. We also show that it suffices to consider strategies in a compact and connected set. Therefore, existence of pure Nash equilibria follows using Brouwer's fixed point theorem [22]. In the second step we prove that iterating the best-response correspondences results in a contraction, which then establishes the uniqueness of the pure Nash equilibrium via the Banach fixed point theorem [22]. The details of the proof are in Appendix A. ■

Denote the expected profits at the unique Nash equilibria by $E[\pi_i^{*,A}]$ for $i = 1, 2$. We will now establish the individual rationality of the sharing contracts that we are analyzing, i.e., the expected profit at equilibrium from sharing (using model A) is always greater than the non-sharing profit.

Proposition 1: Sharing is always individually rational, i.e., $E[\pi_i^{*,A}] \geq E[\tilde{\pi}_i^{*,NS}]$ for $i = 1, 2$.

Proof: Note that for any given (q_1, q_2), the expected profit of service provider 1 (without loss of generality, as a similar argument holds for 2) is given by

$$\begin{aligned}E[\pi_1^A] &= E[\tilde{\pi}_1^{NS}] \\ &+ \alpha p_1 \int_{q_1}^\infty \int_0^{q_2} \min(y - q_1, q_2 - y) f(x, y)dydx \\ &+ (1-\beta)p_2 \int_0^{q_1} \int_{q_2}^\infty \min(q_1 - x, y - q_2) f(x, y)dydx,\end{aligned}$$

where the last two terms are non-negative. This implies that

$$E[\pi_1^A(q_1, q_2)] \geq E[\tilde{\pi}_1^{NS}(q_1)]$$

where we are explicitly indicating the arguments. Setting $q_1 = q_1^{*,NS}$, i.e., the no-sharing optimal purchase, we get

$$\begin{aligned}\max_{q_1} E[\pi_1^A(q_1, q_2)] &\geq E[\pi_1^A(q_1^{*,NS}, q_2)] \\ &\geq E[\tilde{\pi}_1^{NS}(q_1^{*,NS})] = E[\tilde{\pi}_1^{*,NS}]\end{aligned}$$

so that expected profit of the best-response to any given q_2 is at least the maximum expected profit from the non-sharing case. Choosing $q_2 = q_2^{*,A}$ and appealing to Theorem 1, the result then follows. ■

The key point to notice in the proof is that the contracts are structured such that the extra revenue from sharing is non-negative and over and above what the non-sharing scenario provides.

A. The equal prices case

An important sub-case that we will discuss further is when $p_1 = p_2 = p$, i.e., the carriers receive the same per unit demand payoff. This can be justified by assuming that the service provided by the carriers is the same. Under this assumption it is sufficient to consider the cost of spectrum normalized by p. To simplify our analysis, we further assume that the contracts satisfy the following symmetry relationship $\alpha = 1-\beta$, which implies that provider 1 gets α of the revenue from any shared customer regardless of which provider it originates from. With this assumption, we have the following simpler characterization of the Nash equilibrium:

Proposition 2: Under the previous symmetry conditions, the Nash equilibrium $(q_1^{*,A}, q_2^{*,A})$ is the unique solution to the following convex optimization problem,

$$\max_{q_1,q_2 \geq 0} V(q_1, q_2)$$

where

$$\begin{aligned} V(q_1, q_2) := &\int_0^{q_1} \left(\frac{F_1^C(x)}{\alpha} + F_1(x) \right) dx - \frac{c_1 q_1}{\alpha p_1} \\ &+ \int_0^{q_2} \left(\frac{F_2^C(x)}{1-\alpha} + F_2(x) \right) dx - \frac{c_2 q_2}{(1-\alpha) p_1} \\ &- \int_0^{q_1+q_2} F_{D_1+D_2}(x) dx \end{aligned}$$

where $F_{D_1+D_2}(\cdot)$ is the cumulative distribution function of the total demand $D_1 + D_2$.

Proof: It is easily verified that $V(q_1, q_2)$ is jointly concave in (q_1, q_2), and strictly concave when $\alpha \in (0, 1)$, so that existence and uniqueness of optima easily follows. After some algebraic manipulations, it is easy to see that the first-order conditions for equilibrium are the same as setting the gradient of $V(q_1, q_2)$ to 0. From the necessary and sufficient conditions for convex optimization [23], [24], the result then follows. It can also be verified that we have a potential game [25] in this specific setting so we can also appeal to results therein to prove the result. ■

For comparison,the no-sharing case corresponds to maximizing the following (separable) function when $p_1 = p_2 = p$,

$$\begin{aligned} \tilde{V}(q_1, q_2) &= \frac{E[\tilde{\pi}_1^{NS}]}{\alpha p} + \frac{E[\tilde{\pi}_2^{NS}]}{(1-\alpha) p} \\ &= \int_0^{q_1} \frac{F_1^C(x)}{\alpha} dx - \frac{c_1 q_1}{\alpha p} + \int_0^{q_2} \frac{F_2^C(x)}{1-\alpha} dx - \frac{c_2 q_2}{(1-\alpha) p} \end{aligned}$$

The right-hand side of this expression separates into two terms, one for each carrier i. The optimization of the first term corresponds to maximizing the area under the complementary distribution function and above the horizontal line with value $\frac{c_1 q_1}{\alpha p}$, which can be viewed as maximizing a "demand" that exceeds the price $\frac{c_1 q_1}{\alpha p}$. The second term can be interpreted similarly. Comparing this to $V(a_1, q_2)$ in the proposition, we see that this contains two similar terms, in each the "demand" is increased due to the presence of sharing; however, the third term is also needed to avoid double counting this extra demand.

Next we compare the Nash equilibrium with the jointly optimal spectrum purchase that tries to maximize the sum of the expected profits of both the carriers. Note that this is not necessarily the objective of the social planner who would instead like to maximize the total demand served by the carriers subject to their outside option of not sharing; we defer the discussion of under-investment or over-investment until later. The pay-off function for maximizing the expected sum profit is given by

$$\begin{aligned} E[\pi] = &\, p_1 q_1 F_1^C(q_1) + p_1 \int_0^{q_1} x f_1(x) dx - c_1 q_1 \\ &+ p_2 q_2 F_2^C(q_2) + p_2 \int_0^{q_2} y f_2(y) dy - c_2 q_2 \\ &+ p_1 \int_{q_1}^{\infty} \int_0^{q_2} \min(x - q_1, q_2 - y) f(x, y) dy dx \\ &+ p_2 \int_0^{q_1} \int_{q_2}^{\infty} \min(q_1 - x, y - q_2) f(x, y) dy dx. \end{aligned}$$

Here α and β do not appear, since how revenue is shared does not impact the total profit. We have the following result for joint profit maximization problem when $p_1 = p_2 = p$.

Proposition 3: When $p_1 = p_2 = p$, $E[\pi]$ is a jointly concave function of (q_1, q_2). The optimal solution for the sum profit maximization problem is the following: the carrier with the lower cost per unit of spectrum buys spectrum facing the total demand while the other carrier stays out. If the costs are the same, then any partition of the spectrum bought when facing the total demand, between the carriers is optimal.

In other words, the optimal here is generally to funnel all investment to the provider with the lower investment costs. For comparison, in the competitive case both providers will generally invest.

Proof: Without loss of generality we can assume that $p = 1$. Then (after some algebraic manipulations) the relevant partial derivatives of the profit are given by

$$\begin{aligned} \frac{\partial E[\pi]}{\partial q_1} &= 1 - c_1 - F_{D_1+D_2}(q_1 + q_2) \\ \frac{\partial E[\pi]}{\partial q_2} &= 1 - c_2 - F_{D_1+D_2}(q_1 + q_2) \\ \frac{\partial^2 E[\pi]}{\partial q_1^2} &= \frac{\partial^2 E[\pi]}{\partial q_2^2} = \frac{\partial^2 E[\pi]}{\partial q_1 \partial q_2} = -f_{D_1+D_2}(q_1 + q_2) \end{aligned}$$

where $f_{D_1+D_2}(\cdot)$ is the probability distribution function corresponding to the cumulative distribution function $F_{D_1+D_2}(\cdot)$. From the above it is easy to see that $E[\pi]$ is jointly concave in (q_1, q_2), though not necessarily strictly concave. Maximizing $E[\pi]$ subject to non-negativity of q_1 and q_2 is now very easy. Let λ_i be the respective Lagrange multipliers for the non-negativity constraints. If $c_1 = c_2$, then $q_1 + q_2 = F_{D_1+D_2}^{-1}(1 - c_1)$ so that exact solution is not unique, and $\lambda_1 = \lambda_2 = 0$.

However, if $c_1 < c_2$, then $q_2 = 0$ and $q_1 = F^{-1}_{D_1+D_2}(1 - c_1)$ with $\lambda_1 = 0$ and $\lambda_2 = c_2 - c_1$. The case of $c_2 < c_1$ is very similar but with carrier 2 playing the role of carrier 1. In other words, the carrier with the lowest cost purchases spectrum facing the total demand while the other carrier stays out. ∎

B. Co-monotone and Counter-monotone case

In this section we consider extreme cases of the dependence of demand across the carriers. For this we use the theory of copulas [26], which characterizes the possible joint distribution between a pair of random variables in terms of their marginals. It is well known that any joint distribution function satisfies the Fréchet-Hoeffding upper and lower copula bounds, which in a certain sense represents the possible extremes of dependency between the variables. In the two-dimensional case, these correspond to co-monotone and counter-monotone copulas with the co-monotone case corresponding to $D_2 = T(D_1)$ for some increasing function and the counter-monotone case corresponding to $D_2 = T(D_1)$ for some decreasing function. Note that a joint density does not exist in both cases so that Theorem 1 is not applicable. However, intuitively it is clear that these are the extreme-cases scenarios for sharing with the co-monotone case yielding the least benefit from sharing and the counter-monotone case yielding the most benefit. We extend our analysis to these cases in this section and then show via numerical results in the following section that this intuition is correct.

First we consider the co-monotone case. Since the marginal distributions are already known, the function T is uniquely determined and given by $T = F_2^{-1} \circ F_1$ where $\circ$ denotes composition. In this case, the expected profit by sharing for service provider 1 is given by

$$E[\pi_1^{cM}] = E[\pi_1^{NS}] + \tag{1}$$
$$1_{\{T(q_1)\le q_2\}}\alpha p_1 \int_{q_1}^{T^{-1}(q_2)} \min(x - q_1, q_2 - T(x)) f_1(x)dx +$$
$$1_{\{T(q_1)\ge q_2\}}(1-\beta)p_2 \int_{T^{-1}(q_2)}^{q_1} \min(q_1 - x, T(x) - q_2) f_1(x)dx$$

where the superscript cM denotes co-monotone. We then have the following result.

Theorem 2: If $p_2 \ge (1-\alpha)p_1$, $p_1 \ge (1-\beta)p_2$ and $\frac{c_1}{p_1} = \frac{c_2}{p_2}$, then $(q_1^{*,NS}, q_2^{*,NS})$ is a pure Nash equilibrium of the spectrum sharing game with expected profit at the equilibrium being $E[\tilde{\pi}_i^{*,NS}]$ for provider $i = 1, 2$.
Assuming $\frac{c_1}{p_1} = \frac{c_2}{p_2}$ is equivalent to $q_2^{*,NS} = T(q_1^{*,NS})$. Then, since each provider does not receive more revenue from serving a customer of the other provider, the best response of provider i to provider $j = \{1, 2\} \setminus \{i\}$ choosing an investment of $q_j^{*,NS}$, is to also choose $q_i^{*,NS}$.

Proof: See Appendix B for details. ∎

We should point out that our proof does not saying anything about the uniqueness of pure Nash equilibria for the spectrum-sharing game in the co-monotone setting. However, since the co-monotone copula can be approached via copulas with a joint density, e.g., Frank copulas [26], appealing to Theorem 1, we believe that the $(q_1^{*,NS}, q_2^{*,NS})$ equilibrium of Theorem 2 is what is obtained in the limit. Note also that we do not establish the existence of pure Nash equilibria without the additional assumption $\frac{c_1}{p_1} = \frac{c_2}{p_2}$.

Now we consider the counter-monotone case. Again, since the marginal distributions are already known, the (decreasing) function T is uniquely determined and given by $T = F_2^{-1} \circ (1 - F_1) = \left(F_2^C\right)^{-1} \circ F_1$; despite the abuse of notation, the distinction between the two functions is relatively clear and should not cause any confusion. In this case the expected profit by sharing for service provider 1 is given by

$$E[\pi_1^{CM}] = E[\pi_1^{NS}] + \tag{2}$$
$$1_{\{T(q_1)\le q_2\}}(1-\beta)p_2 \int_0^{T^{-1}(q_2)} \min(q_1 - x, T(x) - q_2) f_1(x)dx$$
$$+ 1_{\{T(q_1)\ge q_2\}}\alpha p_1 \int_{T^{-1}(q_2)}^{\infty} \min(x - q_1, q_2 - T(x)) f_1(x)dx$$

with CM standing for counter-monotone. This is considerably harder to analyze. However, we will use this in our discussion for comparison purposes.

IV. Sharing Model B

We consider an alternate sharing model where the carrier accepting traffic from its competitor cannot discriminate between the two types of traffic that it has to carry owing to sharing. Many service providers implement such a policy due to network neutrality concerns. Here we will only consider the independent markets model and, as before, the game between the carriers is to decide on the spectrum investment level so as to maximize individual profit. Once again the contracts are determined by two parameters $\alpha, \beta \in [0, 1]$ that determine the fraction of extra revenue that the service provider with excess demand can serve. Note that there is no sharing when both carriers either have excess spectrum or excess demand. Additionally, when carrier 1 has excess demand, then she serves q_1 amount of traffic and sends the remainder to carrier 2, who then implements the proportional sharing rule; the carrier who faces the two types of demands serves each in proportion to the submitted demand when the total is above the spectrum investment. In this setting the two carriers try to cooperate as much as possible in terms of serving net demand. The expected profit can be rewritten as follows:

$$E[\pi_1^B] = -c_1 q_1 + p_1 q_1 F_1^C(q_1) + p_1 F_2(q_2) \int_0^{q_1} x f_1(x)dx$$
$$+ \int_{q_2}^{\infty}\int_0^{q_1} \min\left(\frac{q_1}{x + y - q_2}, 1\right) \times (p_1 x + (1-\beta)p_2(y - q_2)) f_1(x) f_2(y)dxdy$$
$$+ \int_0^{q_2}\int_{q_1}^{\infty} \min\left(\frac{q_2}{y + x - q_1}, 1\right) \times \alpha p_1 (x - q_1) f_1(x) f_2(y)dxdy$$

where B stands for model B. For simplicity we will assume that the carrier with excess demand transfers the revenue from the amount of traffic served by the carrier with the excess spectrum, i.e., we set $\alpha = \beta = 0$. Going into further details the expected profit of carrier 1 is given by

$$E[\pi_1^B] = -c_1 q_1 + p_1 q_1 F_1^C(q_1) + p_1 F_2(q_2) \int_0^{q_1} x f_1(x) dx + \int_{q_2}^{\infty} \int_0^{q_1} \min\left(\frac{q_1}{x+y-q_2}, 1\right) \times \left(p_1 x + p_2(y-q_2)\right) f_1(x) f_2(y) dx dy$$

Again, the case of $p_1 = p_2 = p$ is easily amenable to analysis; it is possible to construct distributions for which one does not get the concavity properties used in the proof of Theorem 1 when p_1 and p_2 are distinct. Here we get a further simplification to

$$E[\pi_1^B] = (p - c_1) q_1 - p F_2(q_2) \int_0^{q_1} (q_1 - x) f_1(x) dx - p \int_{q_2}^{q_1+q_2} \int_0^{q_1+q_2-y} \left(q_1 + q_2 - x - y\right) f_1(x) f_2(y) dx dy$$

Proposition 4: For sharing model B, if $p_1 = p_2 = p$ and $\alpha = \beta = 0$, then there exists a unique pure Nash equilibrium.

Proof: Taking derivatives in q_1 we get

$$\frac{\partial E[\pi_1^B]}{\partial q_1} = p - c_1 - p F_1(q_1) F_2(q_2) - p \int_{q_2}^{q_1+q_2} F_1(q_1 + q_2 - y) f_2(y) dy$$

$$\frac{\partial^2 E[\pi_1^B]}{\partial q_1^2} = -p f_1(q_1) F_2(q_2) - p \int_{q_2}^{q_1+q_2} f_1(q_1 + q_2 - y) f_2(y) dy$$

$$\frac{\partial^2 E[\pi_1^B]}{\partial q_2 \partial q_1} = -p \int_{q_2}^{q_1+q_2} f_1(q_1 + q_2 - y) f_2(y) dy$$

Note that strict concavity follows and using the same logic as in the proof of Theorem 1 existence and uniqueness of the pure Nash equilibrium follows. ■

With some algebraic manipulations where the independence assumption is critically used, it is possible to show that the expected profit of carrier i in q_i in this model with $p_1 = p_2 = p$ and $\alpha = \beta = 0$ is exactly the same as that of model A with $\alpha = \beta = 0$, i.e., the regime where the carriers that own the spectrum keep the profit from sharing spectrum. Therefore, in both cases the same amount spectrum will be purchased by both carriers and we also get individual rationality for sharing. However, in general, the contracts that we have analyzed need not be individually rational as proportional sharing can result in a service provider dropping some of her traffic, which could result in a net loss of revenue, particularly when the transfer from the competitor is not sufficient to offset the loss. Designing contracts that are provably individually rational for this sharing model is for future research.

V. Discussion

In this section we will explore the consequences of Theorem 1 and compare the resulting equilibrium with the no-sharing scenario. For simplicity we will restrict attention to a symmetric setting where the marginal distributions, the per unit cost of spectrum and the sharing contracts are the same for both service providers; the normalized per unit cost of spectrum will be assumed to be 0.5 in the remainder; note that no-sharing investment level will correspond to the medians of the respective distributions. We will vary the dependence structure between the two markets. As mentioned before, intuitively it is clear that there will be minimal benefits to sharing if the markets are co-monotone, i.e., perfectly dependent which in this case corresponds to the same random variable being chosen for both if the marginals are the same. On the contrary, counter-monotonicity would yield the most benefits as the probability of having excess demand at one service provider and excess spectrum supply at the other provider is maximized. Given the technical requirements of Theorem 1 of the joint distribution possessing a joint density function, we will approach/approximate these two extreme cases via the parameterized Frank copulas [26]. This also allows us to consider the case when the markets are independent.

First we assume the demands follow a Weibull distribution with shape parameter 0.5 and scale parameter 0.5 so that the mean is 1 but the distribution is heavy-tailed (sub-exponential tail). Using the newsvendor model from Section II, the optimal spectrum purchase is $q_i^{*,NS} = \log^2(2)/2 \approx 0.2402$ units. The expected profit of each service provider is 0.0333 units. In Figure 1 we compare the resulting Nash equilibrium spectrum purchase for symmetric contracts as the contract sharing parameters α and β are varied from 0 to 1. In all cases, the expected profit is greater than 0.0333 units so that the carriers will participate in the contract. In general, if the spectrum owner keeps most of the revenue from sharing ($\alpha, \beta < 0.5$), then more spectrum is bought. In addition, when the demands are independent or (approximately) counter monotone even when the spectrum owner keeps a smaller part of the revenue from sharing, there is greater incentive to purchase more spectrum than in the no sharing case. The per carrier spectrum purchase for sum profit maximization is 0.5156, 0.4534 and 0.2554 for the (approximately) counter monotone, independent and (approximately) co-monotone cases, respectively. Therefore, if the spectrum owner gets the bulk of the revenue from sharing, then there is incentive to buy more spectrum than the sum profit maximization strategy.

Next we consider the case of the demands being uniformly distributed in $[0, 2]$. In this case, the service providers each purchase 1 unit of spectrum when there is no sharing. Again, the expected profit from sharing is strictly better than the no sharing case. From Figure 2 it is clear that the carriers buy more spectrum only when the spectrum owner gets a bigger share of the revenue from sharing. The sum profit maximizing strategy in this case is for both carriers to buy 1 unit of spectrum.

Finally, we consider the case of mixed demands where carrier 1 has a demand that is uniform in $[0, 2]$ and carrier 2 has a demand that is Weibull distributed with scale and shape parameter being 0.5. Since the setting is no longer symmetric, the resulting equilibrium is also not symmetric. From Figure 3 we note that the behavior is similar to the first case such that more spectrum is bought by the carriers with sharing even when the spectrum owner does not get most of the revenue from sharing.

The broad conclusions that we can draw from the numerical investigations is that sharing increases the expected profit for the carriers so that it is in their best interests to do so. This is in line with the conclusions of Proposition 1. Note that the specific structure of the contracts determines whether more or less spectrum is purchased with sharing when compared to the no sharing case. In general, if the spectrum owner retains the bulk of the revenue from sharing, then more spectrum is purchased.

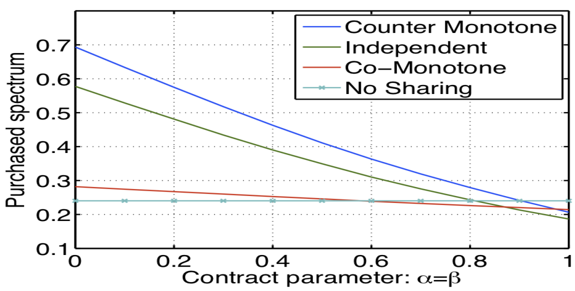

Fig. 1: Comparison of Nash equilibrium to the no sharing scenario for Weibull demand.

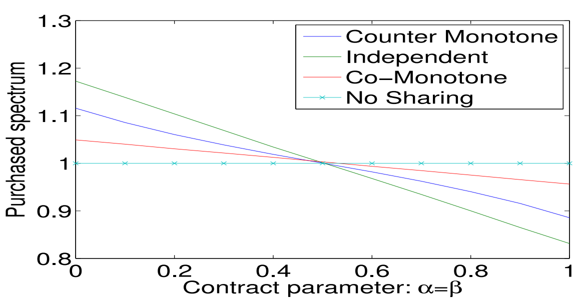

Fig. 2: Comparison of Nash equilibrium to the no sharing scenario for uniform demand.

VI. Conclusion

We analyzed the trade-offs involved in allowing capacity sharing between wireless service providers. For a sharing model where service providers prioritize their own traffic, we demonstrated a family of revenue-sharing contracts that incentivize capacity-sharing despite strategic behavior of the providers. Within this family, we also demonstrated a subset that additionally, achieved the socially desirable goal of servicing more customers. This typically required that the spectrum owner be able to obtain the bulk of the revenue

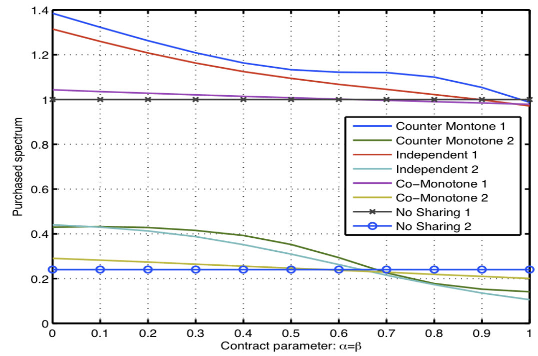

Fig. 3: Comparison of Nash equilibrium to the no sharing scenario for mixed demand.

from sharing. For a proportional sharing model, justified using neutrality concerns, we gave an example of a contract that incentivizes sharing and also results in more customers being serviced. Detailed study of contracts for the proportional sharing model is, however, for future research. Here the contract parameters were exogenously given, an interesting direction of future work is to make these an endogenous part of the model. Also here we focused on a one period model, another potential direction is to consider multiple stages of investing and sharing.

Appendix A
Proof of Theorem 1

For this we start by taking (partial) derivatives of the expected profit in order to characterize a maximizer. We have

$$\frac{\partial E[\pi_1^A]}{\partial q_1} = p_1 F_1^C(q_1) - c_1 - \alpha p_1 \int_0^{q_2} \int_{q_1}^{q_1+q_2-y} f(x,y)dxdy + (1-\beta)p_2 \Big[\int_{q_2}^{q_1+q_2} \int_{q_1+q_2-y}^{q_1} f(x,y)dxdy + \int_{q_1+q_2}^{\infty} \int_0^{q_1} f(x,y)dxdy \Big]$$

$$\frac{\partial^2 E[\pi_1^A]}{\partial q_1^2} = -p_1 f_1(q_1) + \alpha p_1 \int_0^{q_2} f(q_1,y)dy + (1-\beta)p_2 \int_{q_2}^{\infty} f(q_1,y)dy - \alpha p_1 \int_0^{q_2} f(q_1+q_2-y,y)dy - (1-\beta)p_2 \int_{q_2}^{q_1+q_2} f(q_1+q_2-y,y)dy$$

$$\frac{\partial^2 E[\pi_1^A]}{\partial q_2 \partial q_1} = -\alpha p_1 \int_0^{q_2} f(q_1+q_2-y,y)dy - (1-\beta)p_2 \int_{q_2}^{q_1+q_2} f(q_1+q_2-y,y)dy$$

We will discuss a few sub-cases before proving the full result. If $p_1 = p_2$, then we can ensure that $\frac{\partial^2 E[\pi_1^A]}{\partial q_1^2} \leq 0$ with it strictly being negative for all $\alpha, \beta \in [0,1]$. Thus, $E[\pi_1^A]$ is a concave function of q_1 in general, and typically strictly concave. Let us assume the conditions for strict concavity. Then the best response $q_1^*(q_2)$ is unique and a continuous function of q_2. If $\alpha p_1 = (1-\beta)p_2$, then too $E[\pi_1^A]$ is a concave function of q_1. If we insist on the reciprocal condition for carrier 2 to ensure that $E[\pi_2^A]$ is a concave function of q_2, then it is the case that $p_1 = p_2$ and $\alpha + \beta = 1$. Here too strict concavity holds for $\alpha \in [0,1]$. Note that similar conclusions hold for $E[\pi_2^A]$. Finally, consider the second-order partial derivative of the expected profit of carrier 1 in q_1 in more detail. We have

$$\frac{\partial^2 E[\pi_1^A]}{\partial q_1^2} = -p_1 f_1(q_1) + \alpha p_1 \int_0^{q_2} f(q_1,y)dy$$
$$+ (1-\beta)p_2 \int_{q_2}^{\infty} f(q_1,y)dy - \alpha p_1 \int_0^{q_2} f(q_1+q_2-y,y)dy$$
$$- (1-\beta)p_2 \int_{q_2}^{q_1+q_2} f(q_1+q_2-y,y)dy$$
$$= -(1-\alpha)p_1 \int_0^{q_2} f(q_1,y)dy - \alpha p_1 \int_0^{q_2} f(q_1+q_2-y,y)dy$$
$$- (p_1 - (1-\beta)p_2) \int_{q_2}^{\infty} f(q_1,y)dy$$
$$- (1-\beta)p_2 \int_{q_2}^{q_1+q_2} f(q_1+q_2-y,y)dy$$

Therefore, if $p_1 \geq (1-\beta)p_2$, then we again have strict concavity of the expected profit. The corresponding condition for the expected profit of carrier 2 is $p_2 \geq (1-\alpha)p_1$. Thus, (without loss of generality owing to symmetry) if $p_1 \geq p_2$, then there is no restriction on β while we are restricted to $\alpha \geq 1 - \frac{p_2}{p_1}$.

Assuming strict concavity then implies the existence of a pure Nash equilibrium via Brouwer's fixed point theorem [22] in an obvious manner if the supports of F_1 and F_2 are compact and connected. Owing to the $-c_1$ term we can argue that this can extend to non-compact supports as well; irrespective of the value of q_2 we can find an upper bound on q_1 beyond which the first derivative will always be negative. We will now show that $\frac{\partial q_1^*(q_2)}{q_2} < 0$ which will then ensure that $q_2^*(q_1^*(q_2))$ is an increasing function of q_2. The required result holds since q_i^* is the unique solution of $\frac{\partial E[\pi_1^A]}{\partial q_1} = 0$; also note that $\frac{\partial E[\pi_1^A]}{\partial q_1}|_{q_1=0} = p_1\left(1 - \alpha P(D_1 + D_2 \leq q_2)\right) \geq 0$ so that the solution will always be an interior point. . Thus, taking derivatives in q_2 on both sides, we get

$$\frac{\partial^2 E[\pi_1^A]}{\partial q_1^2}\frac{\partial q_1^*(q_2)}{\partial q_2} + \frac{\partial^2 E[\pi_1^A]}{\partial q_2 \partial q_1} = 0$$

Since both the second-order partial derivatives are negative, the conclusion follows. From this we can also write down

$$\frac{\partial q_1^*(q_2)}{\partial q_2} = -\frac{\partial^2 E[\pi_1^A]}{\partial q_2 \partial q_1} / \frac{\partial^2 E[\pi_1^A]}{\partial q_1^2} =$$
$$\frac{-1}{\frac{(1-\alpha)p_1 \int_0^{q_2} f(q_1,y)dy + (p_1-(1-\beta)p_2)\int_{q_2}^{\infty} f(q_1,y)dy}{\alpha p_1 \int_0^{q_2} f(q_1+q_2-y,y)dy - (1-\beta)p_2 \int_{q_2}^{q_1+q_2} f(q_1+q_2-y,y)dy} + 1}$$

From the above, note that the absolute value is at most 1; the value is 1 only when $p_1 = (1-\beta)p_2$ and $\alpha = 1$. A similar form holds for $\frac{\partial q_2^*(q_1)}{\partial q_1}$. Since $\frac{q_2^*(q_1^*(q_2))}{q_2} = \frac{q_2^*(q_1^*(q_2))}{q_1}\frac{q_1^*(q_2)}{q_2}$, it is easy to see that $q_2^*(q_1^*(q_2))$ is a contraction. Thus, uniqueness of the Nash equilibrium follows from the Banach fixed point theorem [22]. From the same theorem it is also clear that iterating the best-response correspondences will yield the unique pure Nash equilibrium.

Appendix B
Proof of Theorem 2

We start by defining a specific increasing function $G(\cdot)$ by considering the condition for equality in the minimum function that appears in the expected profit expression in (1), viz., $x + T(x) = q_1 + q_2$. Let $G(\cdot)$ denote the inverse of $x + T(x)$, which is well defined as $x + T(x)$ is increasing in x. We also note that $\frac{c_1}{p_1} = \frac{c_2}{p_2}$ implies that

$$q_2^{*,NS} = T(q_1^{*,NS}), \text{ and } G\left(q_1^{*,NS} + q_2^{*,NS}\right) = q_1^{*,NS}.$$

Then the expected profit expression can be rewritten as follows

$$E[\pi_1^{cM}] = E[\pi_1^{NS}] + 1_{\{T(q_1)\leq q_2\}}\alpha p_1 \int_{q_1}^{G(q_1+q_2)} (x-q_1)f_1(x)dx$$
$$+ 1_{\{T(q_1)\leq q_2\}}\alpha p_1 \int_{G(q_1+q_2)}^{T^{-1}(q_2)} (q_2 - T(x))f_1(x)dx$$
$$+ 1_{\{T(q_1)\geq q_2\}}(1-\beta)p_2 \int_{T^{-1}(q_2)}^{G(q_1+q_2)} (T(x)-q_2)f_1(x)dx$$
$$+ 1_{\{T(q_1)\geq q_2\}}(1-\beta)p_2 \int_{G(q_1+q_2)}^{q_1} (q_1 - x)f_1(x)dx$$

It can be verified that $T(q_1) \leq q_2$ implies that $q_1 \leq G(q_1 + q_2) \leq T^{-1}(q_2)$ and $T(q_1) \geq q_2$ implies that $T^{-1}(q_2) \leq G(q_1+q_2) \leq q_1$. Taking partial derivatives in q_1 yields the following once we use the definition of $G(\cdot)$

$$\frac{\partial E[\pi_1^{cM}]}{\partial q_1} = \frac{\partial E[\pi_1^{NS}]}{\partial q_1} - 1_{\{T(q_1)\leq q_2\}}\alpha p_1 \int_{q_1}^{G(q_1+q_2)} f_1(x)dx$$
$$+ 1_{\{T(q_1)\geq q_2\}}(1-\beta)p_2 \int_{G(q_1+q_2)}^{q_1} f_1(x)dx$$

If $T(q_1^{*,NS}) = q_2^{*,NS}$, then for $q_2 = q_2^{*,NS}$, the first derivative in q_1 is zero at $q_1 = q_1^{*,NS}$. Let $q_1 < q_1^{*,NS}$, then $T(q_1) \leq T(q_1^{*,NS}) = q_2^{*,NS}$ and $q_1 \leq G(q_1 + q_2^{*,NS}) \leq G(q_1^{*,NS} + q_2^{*,NS}) = q_1^{*,NS}$, and so the ratio of the partial derivative to p_1 is

$$\frac{1}{p_1}\frac{\partial E[\pi_1^{cM}]}{\partial q_1} = F_1^C(q_1) - \frac{c_1}{p_1} - \alpha \int_{q_1}^{G(q_1+q_2^{*,NS})} f_1(x)dx$$
$$= F_1^C(q_1) - F_1^C(q_1^{*,NS}) - \alpha \int_{q_1}^{G(q_1+q_2^{*,NS})} f_1(x)dx$$
$$= \int_{q_1}^{q_1^{*,NS}} f_1(x)dx - \alpha \int_{q_1}^{G(q_1+q_2^{*,NS})} f_1(x)dx \geq 0$$

where we used the definition of $q_1^{*,NS}$. Similarly, if $q_1 > q_1^{*,NS}$, then $T(q_1) \geq T(q_1^{*,NS}) = q_2^{*,NS}$ and $q_1 \geq G(q_1 + q_2^{*,NS}) \geq G(q_1^{*,NS} + q_2^{*,NS}) = q_1^{*,NS}$, and so the ratio of the partial derivative to p_1 is

$$\frac{1}{p_1}\frac{\partial E[\pi_1^{cM}]}{\partial q_1} = F_1^C(q_1) - \frac{c_1}{p_1} + \frac{(1-\beta)p_2}{p_1}\int_{G(q_1+q_2^{*,NS})}^{q_1} f_1(x)dx$$

$$= F_1^C(q_1) - F_1^C(q_1^{*,NS}) + \frac{(1-\beta)p_2}{p_1}\int_{q_1}^{G(q_1+q_2^{*,NS})} f_1(x)dx$$

$$= -\int_{q_1^{*,NS}}^{q_1} f_1(x)dx + \frac{(1-\beta)p_2}{p_1}\int_{q_1}^{G(q_1+q_2^{*,NS})} f_1(x)dx \leq 0$$

where the last inequality necessarily holds if $p_1 \geq (1-\beta)p_2$. Thus, the expected profit is maximized at $q_1^{*,NS}$ as we have shown it to be quasiconcave [27], [28]. The same argument applies to the expected profit of provider 2 where the equivalent condition is $p_2 \geq (1-\alpha)p_1$, and this finishes the proof. It is easy to see that the profit is exactly the profit in the non-sharing case.

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