A simple LP relaxation for the Asymmetric Traveling Salesman Problem

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Abstract. A long-standing conjecture in Combinatorial Optimization is that the integrality gap of the Held-Karp LP relaxation for the Asymmetric Traveling Salesman Problem (ATSP) is a constant. In this paper, we give a simpler LP relaxation for the ASTP. The integrality gaps of this relaxation and of the Held-Karp relaxation are within a constant factor of each other. Our LP is simpler in the sense that its extreme solutions have at most $2n-2$ non-zero variables, improving the bound $3n-2$ proved by Vempala and Yannakakis for the extreme solutions of the Held-Karp LP relaxation. Moreover, more than half of these non-zero variables can be rounded to integers while the total cost only increases by a constant factor.
We also show that given a partially rounded solution, in an extreme solution of the corresponding LP relaxation, at least one positive variable is greater or equal to 1/2.

Key words: ATSP, LP relaxation

1 Introduction

The Traveling Salesman Problem (TSP) is a classical problem in Combinatorial Optimization. In this problem, we are given an undirected or directed graph with nonnegative costs on the edges, and we need to find a Hamiltonian cycle of minimum cost. A Hamiltonian cycle is a simple cycle that covers all the nodes of the graph. It is well known that the problem is in-approximable for both undirected and directed graphs. A more tractable version of the problem is to allow the solution to visit a vertex/edge more than once if necessary. The problem in this version is equivalent to the case when the underlying graph is a complete graph, and the edge costs satisfy the triangle inequality. This problem is called the metric-TSP, more specifically Symmetric-TSP (STSP) or Asymmetric-TSP (ATSP) when the graph is undirected or directed, respectively. In this paper, we consider the ATSP.

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Notation. In the rest of the paper, we need the following notation. Given a directed graph $G = (V, E)$ and a set $S \subset V$, we denote the set of edges going in and out of S by $\delta^+(S)$ and $\delta^-(S)$, respectively. Let x be a non-negative vector on the edges of the graph G, the in-degree or out-degree of S (with respect to x) is the sum of the value of x on $\delta^+(S)$ and $\delta^-(S)$. We denote them by $x(\delta^+(S))$ and $x(\delta^-(S))$.

An LP relaxation of the ATSP was introduced by Held and Karp [9] in 1970. It is usually called the Held-Karp relaxation. Since then it has been an open problem to show whether this relaxation has a constant integrality gap. The Held-Karp LP relaxation can have many equivalent forms, one of which requires a solution $x \in \mathbb{R}_+^{|E|}$ to satisfy the following two conditions: *i)* the in-degree and the out-degree of every vertex are at least 1 and equal to each other, and *ii)* the out-degree of every subset $S \subset V - \{r\}$ is at least 1, where r is an arbitrary node picked as a root. Fractional solutions of this LP relaxation are found to be hard to round because of the combination of the degree conditions on each vertex and the connectivity condition. A natural question is to relax these conditions to get an LP whose solutions are easier to round. In fact, when these conditions are considered separately, their LP forms integral polytopes, thus the optimal solution can be found in polynomial time. However, the integrality gap of these LPs with respect to the integral solutions of the ATSP can be arbitrarily large. Another attempt is to keep the connectivity condition and relax the degree condition on each vertex. It is shown recently by Lau et al. [15] that one can find an integral solution whose cost is at most a constant times the cost of the LP described above, furthermore it satisfies the connectivity condition and violates the degree condition at most a constant. The solution found does not satisfy the balance condition on the vertices, and such a solution can be very far from a solution of the ATSP.

Generally speaking, there is a trade-off in writing an LP relaxation for a discrete optimization problem: between having "simple enough" LP to round and a "strong enough" one to prove an approximation guarantee. It is a major open problem to show how strong the Held-Karp relaxation is. And, as discussed above, it seems that all the simpler relaxations can have arbitrarily big integrality gaps. In this paper, we introduce a new LP relaxation of the ATSP which is as strong as the Help-Karp relaxation up to a constant factor, and is simpler. Our LP is simpler in the sense that an extreme solution of this LP has at most $2n-2$ non-zero variables, improving the bound $3n-2$ on the extreme solutions of the Held-Karp relaxation. Moreover, out of such $2n-2$ variables, at least n can be rounded to integers. This result shows that the integrality gap of the Held-Karp relaxation is a constant if and only if our *simpler* LP also has a constant gap.

The new LP. The idea behind our LP formulation is the following. Consider the Held-Karp relaxation in one of its equivalent forms:

$$\begin{aligned} \min \quad & c_e x_e \\ \text{Sbjt:} \quad & x(\delta^+(S)) \geq 1 \; \forall S \subset V - \{r\} \quad \textit{(Connectivity condition)} \\ & x(\delta^+(v)) = x(\delta^-(v)) \; \forall v \in V \quad \textit{(Balance condition)} \\ & x_e \geq 0. \end{aligned} \tag{1}$$

Our observation is that because of the balance condition in the LP above, the in-degree $x(\delta^+(S))$ is equal to the out-degree $x(\delta^-(S))$ for every set S . If one can guarantee that the ratio between $x(\delta^+(S))$ and $x(\delta^-(S))$ is bounded by a constant, then using a theorem of A. J. Hoffman [10] about the condition for the existence of a circulation in a network, we can still get a solution satisfying the balance condition for every node with only a constant factor loss in the total cost. The interesting fact is that when allowed to relax the balance condition, we can combine it with the connectivity condition in a single constraint. More precisely, consider the following fact. Given a set $S \subset V - \{r\}$, the balance condition implies $x(\delta^+(S)) - x(\delta^-(S)) = 0$, and the connectivity condition is $x(\delta^+(S)) \geq 1$. Adding up these two conditions, we have:

$$2x(\delta^+(S)) - x(\delta^-(S)) \geq 1.$$

Thus we can have a valid LP consisting of these inequalities for all $S \subset V - \{r\}$ and two conditions on the in-degree and out-degree of r. Observe that given a vector $x \geq 0$, the function $f(S) = 2x(\delta^+(S)) - x(\delta^-(S))$ is a submodular function, therefore, we can apply the uncrossing technique as in [11] to investigate the structure of an extreme solution. We introduce the following LP:

$$\begin{aligned} \min \quad & c_e x_e \\ \text{Subject to:} \quad & 2x(\delta^+(S)) - x(\delta^-(S)) \geq 1 \; \forall S \subset V - \{r\} \\ & x(\delta^+(r)) = x(\delta^-(r)) = 1 \\ & x_e \geq 0. \end{aligned} \tag{2}$$

This LP has exponentially many constraints. But because $2x(\delta^+(S)) - x(\delta^-(S))$ is a submodular function, the LP can be solved in polynomial time via the ellipsoid method and a subroutine to minimize submodular setfunctions.

Our results. It is not hard to see that our new LP (2) is weaker than the Held-Karp relaxation (1). In this paper, we prove the following result in the reverse direction. Given a feasible solution x of (2), in polynomial time we can find a solution y feasible to (1) on the support of x such that the cost of y is at most a constant factor of the cost of x. Furthermore, if x is integral then y can be chosen to be integral as well. Thus, given an integral solution of (2) we can find a Hamiltonian cycle of a constant approximate cost. This also shows that the integrality gaps of these two LPs are within a constant factor of each other. In section 3, we show

that our new LP is simpler than the Held-Karp relaxation. In particular, we prove that an extreme solution of the new LP has at most $2n - 2$ non-zero variables, improving the bound $3n - 2$ proved by Vempala and Yannakakis [17] for the extreme solutions of the Held-Karp relaxation. We then show how to round at least n variables of a fractional solution of (2) to integers. And finally, we prove the existence of a big fractional variable in an extreme point of our LP in a partially rounded instance.
Note that one can have a more general LP relaxation by adding the Balance Condition and the Connectivity Condition in (1) with some positive coefficient (a, b) to get: $(a+b)x(\delta^+(S)) - bx(\delta^-(S)) \geq a$. All the results will follow, except that the constants in these results depend on a and b. One can try to find a and b to minimize these constants. But, to keep this extended abstract simple, we only consider the case where $a = b = 1$.

Related Work. The Asymmetric TSP is an important problem in Combinatorial Optimization. There is a large amount of literature on the problem and its variants. See the books [8], [16] for references and details. A natural LP relaxation was introduced by Held-Karp [9] in 1970, and since then many works have investigated this LP in many aspects. See [8] for more details. Vempala and Yannakakis [17] show a sparse property of an extreme solution of the Held-Karp relaxation. Carr and Vempala [4] investigated the connection between the Symmetric TSP (STSP) and the ATSP. They proved that if a certain conjecture on STSP is true then the integrality gap of this LP is bounded by 4/3. Charikar et al. [3] later refuted this conjecture by showing a lower bound of 2 for the integrality gap of the Held-Karp LP, this is currently the best known lower bound. On the algorithmic side, a $\log_2 n$ approximation algorithm for the ATSP was first proved by Frieze et al. [6]. This ratio is improved slightly in [2], [12]. The best ratio currently known is $0.842 \log_2 n$ [12].
Some proofs of our results are based on the uncrossing technique, which was first used first by László Lovász [5] in a mathematical competition for university students in Hungary. The technique was later used successfully in Combinatorial Optimization. See the book [13] for more details. In Approximation Algorithms, the uncrossing technique was applied to the the generalized Steiner network problem by Kamal Jain [11]. And it is recently shown to be a useful technique in many other settings [7, 14, 15].

2 The integrality gaps of the new LP and of the Held-Karp relaxation are essentially the same.

In this section, we prove that our LP and the Held-Karp relaxation have integrality gaps within a constant factor of each other. We also show that given an integral solution of the new LP (2), one can find a Hamilton cycle while only increasing the cost by a constant factor.

Theorem 1. *Given a feasible solution of the Held-Karp relaxation (1), we can find a feasible solution of (2) with no greater cost. Conversely, if x is a solution of (2) then there is a feasible solution y of (1) on the support* *of x, whose cost is at most a constant times the cost of x. Moreover, such a y can be found in polynomial time, and if x is integral then y can also be chosen to be an integral vector.*

The second part of this theorem is the technical one. As discussed in the introduction, at the heart of our result is the following theorem of Alan Hoffman [10] about the condition for the existence of a circulation in a network.

Lemma 1 (Hoffman). *Consider the LP relaxation of a circulation problem on a directed graph $G = (V, E)$ with lower and an upper bounds $l_e \leq u_e$ on each edge $e \in E$:*

$$\begin{aligned} & x(\delta^+(v)) = x(\delta^-(v)) \;\; \forall v \in V \\ & l_e \leq x_e \leq u_e \;\; \forall e \in E. \end{aligned} \tag{3}$$

The LP is solvable if and only if for every set S :

$$\sum_{e \in \delta^+(S)} l_e \leq \sum_{e \in \delta^-(S)} u_e.$$

Furthermore, if l_e, u_e are integers, then the solution can be chosen to be integral. □

Given a solution x of our new LP, we use it to set up a circulation problem. Then, using Lemma 1, we prove that there exists a solution y to this circulation problem. And this vector is a feasible solution of the Help-Karp relaxation. Before proving Theorem 1, we need the following lemmas:

Lemma 2. *For a solution x of (2) and every set $S \subsetneq V$, the in-degree $x(\delta^+(S))$ and the out-degree $x(\delta^-(S))$ are at least $\frac{1}{3}$.*

Proof. Because of symmetry, we assume that $r \notin S$. Since x is a solution of (2), we have: $2x(\delta^+(S)) - x(\delta^-(S)) \geq 1$. This implies $x(\delta^+(S)) \geq \frac{1}{2} + \frac{1}{2}x(\delta^-(S)) > \frac{1}{3}$.
We now prove that $x(\delta^-(S)) \geq \frac{1}{3}$. If $S = V - \{r\}$, then because of the LP (2) the out-degree of S is 1, which is of course greater than $\frac{1}{3}$. Now, assume S is a real subset of $V - \{r\}$, let $T = V - \{r\} - S \neq \emptyset$.
To make the formula easy to follow, we use the following notation. Let α, β be the total value of the edges going from S to r and T respectively. See Figure 1. Thus, the out-degree of S is $x(\delta^-(S)) = \alpha + \beta$. We denote the total value of the edges going from r to S by a, and the total value of edges from T to S by b. Due to (2), the in-degree and out-degree of r is 1, therefore the total value of edges from r to T is $1 - a$ and from T to r is $1 - \alpha$.
Now, from $2x(\delta^+(T)) - x(\delta^-(T)) \geq 1$, we have $2(1 - a + \beta) - (1 - \alpha + b)$. Therefore

$$2\beta + \alpha \geq 2a + b.$$

And $2x(\delta^+(S)) - x(\delta^-(S)) \geq 1$ is equivalent to $2(a + b) - (\alpha + \beta) \geq 1$, which implies

$$(a + b) \geq \frac{(\alpha + \beta) + 1}{2}.$$

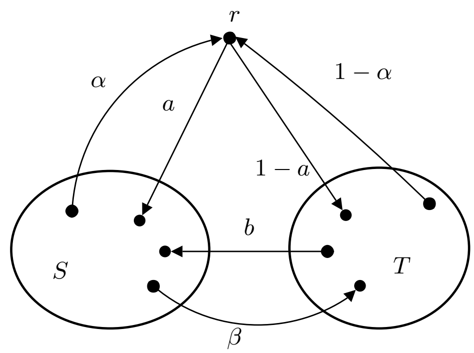

Fig. 1. Out-degree and in-degree of the set S

Combine these two inequalities:

$$2\beta + \alpha \geq 2a + b \geq a + b \geq \frac{\alpha + \beta + 1}{2}.$$

Thus we have $2\beta + \alpha \geq \frac{\alpha+\beta+1}{2}$. From this, $4\beta + 2\alpha \geq \alpha + \beta + 1$ and $3\beta + \alpha \geq 1$. Hence, $3(\beta + \alpha) \geq 3\beta + \alpha \geq 1$. Therefore

$$\alpha + \beta \geq \frac{1}{3}.$$

This inequality is what we need to prove. □

The next lemma shows that for any S, the ratio between its out-degree and in-degree is bounded by a constant.

Lemma 3. *Given a solution x of (2), for any $S \subsetneq V$,*

$$\frac{1}{8}x(\delta^-(S)) \leq x(\delta^+(S)) \leq 8x(\delta^-(S)).$$

Proof. Because of symmetry, we can assume that $r \notin S$. From the inequality $2x(\delta^+(S)) - x(\delta^-(S)) \geq 1$ we have:

$$x(\delta^-(X)) < 2x(\delta^+(X)). \text{ Therefore } \frac{1}{8}x(\delta^-(S)) \leq x(\delta^+(S)).$$

To show the second inequality, we observe that when $S = V - \{r\}$, its out-degree is equal to its in-degree, thus we can assume that S is a real subset of $V - \{r\}$. As in the previous lemma, let $T = V - S - \{r\} \neq \emptyset$. We then apply the inequality $2x(\delta^+(T)) - x(\delta^-(T)) \geq 1$ to get the desired inequality.

First, observe that S and T are almost complements of each other, except that there is a node r with in and out degrees of 1 outside S and T. Thus, the out-degree of S is almost the same as the in-degree of T and vice versa. More precisely, using the same notation as in the previous lemma, one has $x(\delta^+(S)) - x(\delta^-(T)) = a - (1 - \alpha) \leq 1$. Therefore $x(\delta^+(S)) \leq x(\delta^-(T)) + 1$.

By symmetry, we also have: $x(\delta^+(T)) \leq x(\delta^-(S)) + 1$.
Now, $2x(\delta^+(T)) - x(\delta^-(T)) \geq 1$ implies:

$$1 + x(\delta^-(T)) \leq 2x(\delta^+(T)).$$

Using the relations between the in/out-degrees of S and T, we have the following:

$$x(\delta^+(S)) \leq 1 + x(\delta^-(T)) \leq 2x(\delta^+(T)) \leq 2(x(\delta^-(S)) + 1).$$

But because of the previous lemma, $x(\delta^-(S)) \geq \frac{1}{3}$. Therefore

$$x(\delta^+(S)) \leq 2(x(\delta^-(S)) + 1) \leq 8x(\delta^-(S)).$$

This is indeed what we need to prove. □

Note: We believe the constant in this lemma can be reduced if we use a more careful analysis.
We are now ready to prove our main theorem:

Proof (Proof of Theorem 1). First, given a solution y, if $y(\delta^+(r)) = y(\delta^-(r)) = 1$, then y is also a feasible solution of (2). When $y(\delta^+(r)) = y(\delta^-(r)) \geq 1$, we can short-cut the fractional tour to get the solution satisfying the degree constraint on r: $y(\delta^+(r)) = y(\delta^-(r)) = 1$ without increasing the cost. This solution is a feasible solution of (2).
We now prove the second part of the theorem. Given a solution x of (2), consider the following circulation problem:

$$\begin{aligned} \min\ & c_e y_e \\ \textit{sbt.}\ & y(\delta^+(v)) = y(\delta^-(v)) \forall v \in V \\ & 3x_e \leq y_e \leq 24x_e. \end{aligned}$$

For every set $S \subset V$, Lemma 2 states that the ratio between its in-degree and its out-degree is bounded by 8. Therefore

$$\sum_{e \in \delta^+(S)} 3x_e \leq \sum_{e \in \delta^-(S)} 24x_e.$$

Using Lemma (1), the above LP has a solution y, and y can be chosen to be integral if x is integral. We need to show that y is a feasible solution of the Held-Karp relaxation. y satisfies the Balance Constraint on every node, thus we only need to show the Connectivity Condition. Because $y \geq 3x$, for every cut S we have :

$$y(\delta^+(S)) \geq 3x(\delta^+(S)) \geq 1.$$

The last inequality comes from Lemma 2. We have shown that given a feasible solution x of the new LP, there exists a feasible solution of the Held-Karp relaxation 1 whose cost is at most 24 times the cost of x. This completes the proof of our theorem. □

3 Rounding an extreme solution of the new LP

In this section, we show that an extreme solution of our LP contains at most $2n-2$ non-zero variables (Theorem 2). And at least n variables of this solution can be rounded to integers (Theorem 3). Finally, given a partially rounded solution, let x be an extreme solution of the new LP for this instance. We show that among the other positive variables, there is at least one with a value greater or equal to 1/2 (Theorem 4).

Theorem 2. *The LP (2) can be solved in polynomial time, and an extreme solution has at most* $2n-2$ *non-zero variables.*

Proof. First, observe that given a vector $x \geq 0$, $f_x(S) = 2x(\delta^+(S)) - x(\delta^-(S))$ is a submodular function. To prove this, one needs to check that $f_x(S) + f_x(T) \geq f_x(S \cup T) + f_x(S \cap T)$. Or more intuitively: $f_x(S) = x(\delta^+(S)) + (x(\delta^+(S)) - x(\delta^-(S)))$ is a sum of two submodular functions, thus f_x is also a submodular function.
The constraints in our LP is $f_x(S) \geq 1 \forall S \subset V - \{r\}$ and $x(\delta^+(r)) = x(\delta^-(r)) = 1$. Thus with a subroutine to minimize a submodular function, we can decide whether a vector x is feasible to our LP, and therefore the LP can be solved in polynomial time by the ellipsoid method.
Now, assume x is an extreme solution. Let S, T be two tight sets, i.e., $f_x(S) = f_x(T) = 1$. Then, it is not hard to see that if $S \cup T \neq \emptyset$ then both $S \cup T$ and $S \cap T$ are tight. Furthermore, the constraint vectors corresponding to $S, T, S \cup T, S \cap T$ are dependent. Now, among all the tight sets, take the maximal laminar set family. The constraints corresponding to these sets span all the other tight constraints. Thus x is defined by 2 constraints for the root node r and the constraints corresponding to a laminar family of sets on $n-1$ nodes, which contains at most $2(n-1)-1$ sets. However, the constraint corresponding to the set $V - r$ is dependent on the two constraints of the node r, therefore we have at most $2 + 2(n-1) - 1 - 1 = 2n - 2$ independent constraints. This shows that x has at most $2n-2$ non-zero variables. □

We prove the next theorem about rounding at least n variables of a fractional solution of our new LP (2).

Theorem 3. *Given an extreme solution* x *of (2), we can find a solution* $\tilde{x}$ *on the support of* x*. Thus* $\tilde{x}$ *contains at most* $2n-2$ *non-zero edges such that it satisfies the constraint* $2\tilde{x}(\delta^+(S)) - \tilde{x}(\delta^-(S)) \geq 1 \ \forall S \subset V - \{r\}$*, and it has at least n non-zero integral variables. Furthermore, the cost of* $\tilde{x}$ *is at most a constant times the cost of* x*.*

Proof. x is a solution of (2). Due to Theorem 1, on the support of x, there exists a solution y of (1) whose cost is at most a constant times the cost of x. Because y satisfies $y(\delta^+(v)) = y(\delta^-(v)) \geq 1$ for every $v \in V$, y is a fractional cycle cover on the support of x. Recall that a cycle cover on directed graph is a Eulerian subgraph (possibly with parallel edges) covering all the vertices. However, we can find an integral cycle cover in a directed graph whose cost is at most the cost of a fractional solution. Let z be such an integral solution. Clearly, z has at least n non-zero

variables, and the cost of z is at most the cost of y which is at most a constant times the cost of x.
Next consider the solution $w = x + \frac{3}{2}z$. For every edge e where $z_e > 0$, we have $w_e = x_e + \frac{3}{2}z_e > \frac{3}{2}$. Round w_e to the closest integer to get the solution $\tilde{x}$. Clearly, $\tilde{x}$ has at most $2n-2$ non-zero variables and at least n non-zero integral variables. We will show that the cost of $\tilde{x}$ is at most a constant times the cost of x, and that $\tilde{x}$ satisfies $2\tilde{x}(\delta^+(S)) - \tilde{x}(\delta^-(S)) \geq 1 \ \forall S \subset V - \{r\}$.
Rounding each w_e to the closest integer will sometimes cause an increase in w_e of at most $1/2$. But, because we only round the value w_e when the corresponding $z_e \geq 1$, and note that z is an integral vector, the total increase is at most half the cost of z which is at most a constant times the cost of x.
Consider a set $S \subset V - \{r\}$. We have $2x(\delta^+(S)) - x(\delta^-(S)) \geq 1$. Let k be the total value of the edges of z going out from S, that is $k = z(\delta^+(S)) = z(\delta^-(S))$. This is true because z is a cycle cover. Hence, when adding $w := x + \frac{3}{2}z$, we have:

$$2w(\delta^+(S)) - w(\delta^-(S)) = 2x(\delta^+(S)) - x(\delta^-(S)) + \frac{3}{2}(2z(\delta^+(S)) - z(\delta^-(S))).$$

Therefore,

$$2w(\delta^+(S)) - w(\delta^-(S)) \geq 1 + \frac{3}{2}k. \tag{4}$$

Now, $\tilde{x}$ is a rounded vector of w on the edges where z is positive. For the set S, there are at most $2k$ such edges, at most k edges going out and k edges coming in. Rounding each one to the closest integer will sometimes cause a change at most $\frac{1}{2}$ on each edge, and thus causes the change of $2w(\delta^+(S)) - w(\delta^-(S))$ in at most $k(2.\frac{1}{2} - (-\frac{1}{2})) = \frac{3}{2}k$. But, because of (4), we have :

$$2\tilde{x}(\delta^+(S)) - \tilde{x}(\delta^-(S)) \geq 1$$

which is what we need to show. □

Our last theorem shows that there always exists a "large" variable in an extreme solution in which some variables are assigned fixed integers.

Theorem 4. *Consider the following LP which is the corresponding LP of (2) when some variables $x_e, e \in F$ are assigned fixed integral values. $x_e = a_e \in \mathbb{N}$ for $e \in F$.*

$$\begin{aligned}
\min \quad & c_e x_e \\
\textit{sbt.} \quad & 2x(\delta^+(S)) - x(\delta^-(S)) \geq 1 \ \forall S : r \notin S. \\
& x(\delta^+(r)) = r_1 \\
& x(\delta^-(r)) = r_2 \ \ (r_1, r_2 \in \mathbb{N}) \\
& x_e = a_e \ \forall e \in F \\
& x_e \geq 0.
\end{aligned} \tag{5}$$

Given an extreme solution x of this LP, let $H = \{e \in E - F | x_e > 0\}$. If $H \neq \emptyset$, then there exists an $e \in H$ such that $x_e \geq \frac{1}{2}$.

Proof. Let $\mathcal{L}$ be the laminar set family whose corresponding constraints together with two constraints on the root node r determine the value of $\{x_e|e \in H\}$ uniquely. As we have seen in the proof of Theorem 2, one can see that such an $\mathcal{L}$ exists, and $|\mathcal{L}|$ is at least $|H|$. Assume all the values in $\{x_e|e \in H\}$ is less than a half. We assign one token to each edge in H. If we can redistribute these tokens to the sets in $\mathcal{L}$ and the constraints on the root r such that each constraint gets at least 1 token, but at the end there are still some tokens remaining, we will get a contradiction to prove the theorem.

We apply the technique used in [11] and the other recent results [14], [15], [1]. For each $e \in H$, we distribute a fraction $1 - 2x_e$ of the token to the head of the edge, a fraction x_e of the token to the tail and the remaining x_e to the edge itself. See Figure 2. Because $0 < x_e < \frac{1}{2}$, all of these values are positive. Given a set S in $\mathcal{L}$, we now describe the set of tokens assigned to this set. First, we use the following notation: for a set T, let $E(T)$ be the set of edges in $\{x_e|e \in E - F\}$ that have both endings in T. Now, let $S_1, .., S_k \in \mathcal{L}$ be the maximal sets which are real subsets of S. The set of tokens that S gets is all the tokens on the edges in $E(S) - (E(S_1) \cup ... \cup E(S_k))$ plus the tokens on the vertices in $S - (S_1 \cup ... \cup S_k)$. Clearly, no tokens are assigned to more than one set. The constraint on the in-degree of r gets all the tokens on the heads of edges going into r, and the constraint on the in-degree of r gets all the tokens on the tails of edges going out from r.

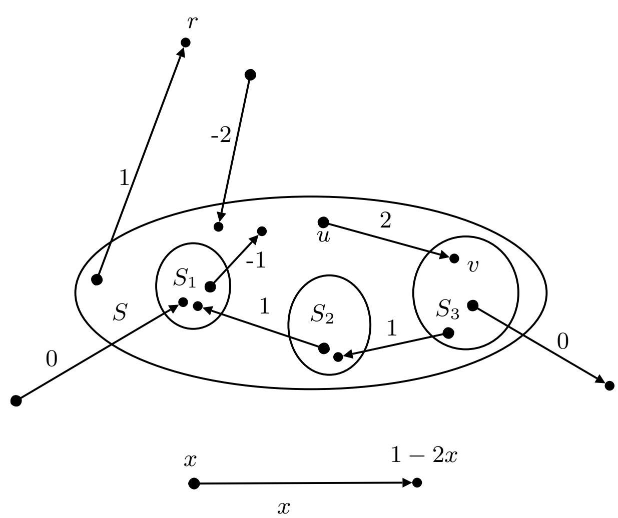

Fig. 2. Tokens distributed to S

Consider now the equalities corresponding to the set $S, S_1, ..., S_k$. If we add the equalities of $S_1, S_2, ..., S_k$ together and subtract the equality on the set S we will get an linear equality on the variables $\{x_e | e \in H\}$:

$$\sum_{e \in H} \alpha_e x_e = \text{ an integer number.}$$

It is not hard to calculate α_e for each type of e. For example, if e connects S_i and S_j , $i \neq j$ then $\alpha_e = 1$, if e connects from vertex outside S to a vertex in $S - (S_1 \cup ... \cup S_k)$ then $\alpha_e = -2$, etc. See Figure 2 for all other cases.

On the other hand, if we calculate the amount of tokens assigned to the set S, it also has a linear formula on $\{x_e | e \in H\}$:

$$\sum_{e \in H} \beta_e x_e + \text{ an integer number.}$$

We can also calculate the coefficient β_e for every e. For example, if e connects S_i and S_j , $i \neq j$ then the edge e is the only one that gives an amount of tokens which is a function of x_e, and it is exactly x_e. Thus $\beta_e = 1$. Consider another example, $e = u \to v$ where $u \in S - (S_1 \cup ... \cup S_k)$ and $v \in S_3$. See Figure 2. Then only the amounts of tokens on the edge uv and the node u depend on x_e. On the edge uv, it is x_e and, on the node u, it is x_e plus a value not depending on x_e. Thus $\beta_e = 2$ in this case.

It is not hard to see that the coefficient $\alpha_e = \beta_e \forall e \in H$. Thus the amount of tokens S gets is an integer number, and it is positive, thus it is at least 1. Similarly, one can show that this fact also holds for the constraints on the root node r.

We now show that there are some tokens that were not assigned to any set. Consider the biggest set in the laminar set family $\mathcal{L}$, it has some non-zero edges going in or out but the tokens on this edge is not assigned to any constraint. This completes the proof. $\square$

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