# The Allocation of Indivisible Objects via Rounding

Thành Nguyen & Rakesh V. Vohra
Department of Managerial Economics and Decision Sciences
Kellogg School of Management
Northwestern University*

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**Abstract**

The problem of allocating indivisible objects arises in the allocation of courses, spectrum licenses, landing slots at airports and assigning students to schools. This paper proposes a technique for making such allocations that is based on rounding a fractional allocation. Under the assumption that no agent wants to consume more than $k$ items, the rounding technique can be interpreted as giving agents lotteries over approximately feasible integral allocations that preserve the ex-ante efficiency and fairness and asymptotically strategy-proof properties of the initial fractional allocation. The integral allocations are only approximately feasible in the sense that upto $k-1$ more units than the available supply of any good is allocated.

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# 1 Introduction

The problem of allocating indivisible objects to agents without the use of transfers is important and widespread. The literature dates back to at least Shapley and Scarf [1974] and Hylland and Zeckhauser [1979]. More recent applications include student assignment (Abdulkadiroglu and Sönmez [2003]) and course allocation (Budish [2011]). In these cases indivisible objects (schools in the first and courses in the second) must be allocated to students without the use of transfers in a manner that is fair, incentive compatible and efficient. Mechanisms for identifying social welfare maximizing allocations (subject to fairness and incentive compatability) in these settings must do two things simultaneously. First, elicit information about each agents preferences which is privately held. The inability to use transfers makes elicitation difficult. As noted by Budish [2012], one reason is the absence of a numeraire good, like money, that can be used to discourage agents from claiming an excessively large utility for their most preferred bundle of objects. Second, because of the large numbers of agents and distinct types of goods involved, do so in a way that is computationally efficient.

There are two basic classes of methods that have been studied in the literature. The first is the Probabilistic Serial mechanism (Bogomolnaia and Moulin [2001] and some of its generalizations (e.g. Peivandi [2013]). These mechanisms use only *ordinal* information about agents preferences as input and return lotteries over allocations that satisfy ordinal only notions of fairness and efficiency. In settings where there are a large number of agents, these mechanisms are approximately strategy-proof in that no agent can materially affect the outcome by misreporting their ordinal preferences. They are, however, not computationally efficient.

The second class of methods endows agents with an artificial currency and allows them to trade. One approach, called "Competitive Equilibrium from Equal Incomes" gives each agent the same amount of an artificial currency; and then sets a market clearing price for each good assuming it is *divisible*. Each agent is then allocated their most preferred bundle from their budget set. As the goods are assumed divisible, the bundles are 'fractional' and are interpreted as lotteries over possible outcomes. An example of this approach is the mechanism proposed by Hylland and Zeckhauser [1979] which works in a setting where each agent wants no more than one good and has preferences that can be represented by a von Neumann-Morgenstern utility function. Recently, Budish [2011] generalized this approach to the case where agents have preferences over bundles by eliciting information about marginal rates of substitution across individual objects. These mechanisms return allocations satisfying a notion of economic efficiency weaker than social welfare maximization, namely Pareto optimality. These methods, because they require the determination of market clearing price, i.e., fixed point, are also computationally inefficient.

This paper offers a computationally efficient approach to finding allocations that are social-welfare maximizing subject to constraints on fairness and incentive compatibility.[1] In particular, we consider a multi-unit assignment problem similar to Budish [2011], and assume the following[2].

1. The size of bundles that each agents assign positive utility to is small compared to the size of the market (measured in terms of the number of agents or goods).

2. The number of agents and supply of goods is large.

Our main idea is to formulate the problem of finding an allocation of objects to agents so as to maximize social welfare (or other similar objective) subject to (interim) envy-freeness as an integer program. An allocation is envy-free if each prefers the bundle assigned to them to any bundled assigned to another agent. Envy-freeness is usual interpreted as a notion of fairness. However, as we show, when the economy gets large, envy-freeness actually implies strategy-proofness. Thus we obtain fairness and incentive compatibility at the same time for large economies.

The integer program we formulate is NP-hard. However, we solve its linear relaxation instead. Subsequently, we use the iterative rounding method (IRM) (Lau et al. [2011]) to construct a lottery over "almost feasible" integer solutiuons from this fractional solution. Our main result is that the mechanism that outputs this lottery is fair, "almost" efficient and strategy-proof.

Our approach has two advantages. First, it allows one to consider stronger notions of efficiency when preferences are cardinal, for example, maximizing social welfare. The previous literature only considers Pareto optimality or ordinal efficiency. Our approach is also more flexible in that, for example, one can replace the objective function of maximizing the sum of agents' utilities by maximizing any weighted sum of allocations. Second, our method is computationally efficient. It does not require the computation of a general equilibrium as in Peivandi [2013], Hylland and Zeckhauser [1979] and Budish [2011]. We only need to solve a polynomial (in the number of agents and number of goods) number of linear programs of polynomial size.

[1]Thus, we assume that agents have cardinal utilities denominated on a common scale.

[2]These conditions hold in several settings. For instance, in course allocation, the number of students and available courses is large relative to the number of courses a student can take each semester/quarter.

Our approach also applies to different types of problems. For example, in a setting with unit demand and side constraints, the linear programing approach naturally extends and give similar results. Unit demand with side constraints settings have important applications, particularly in school choice problems. Here, no student can be assigned to more than one school. However, the assignment of students to schools must satisfy additional constraints regarding the representation of minorities, see for example Ehlers et al. [2011]. This application also naturally satisfies the our assumption in that the number of such side constraints is small relative to the capacity constraints on each school.

In the following, we describe our results for the Combinatorial Assignment Problem in more details. Other applications including school choice with side constraints; approximate Walrasian prices and combinatorial auctions will be discussed in Section 5.

## Combinatorial Assignment Problem

In the combinatorial assignment problem we have a set $G$ of goods and for each $j \in G$ we have a supply of $s_j$ units. Let $N$ be the set of agents. Each agent's preferences can be represented by a von-Neuman Morgenstern utility function over subsets of $G$. A feasible assignment of goods to agents ensures that each agent receives at most one unit of each good and the total amount of any good allocated does not exceed its supply. One application of the combinatorial assignment problem is course allocation. $G$ is the set of courses and $s_j$ is the number of seats in course $j \in G$.

To define what it means for the size of a bundle that an agent requests to be small, denote by $u(S)$ an agent's von-Neuman Morgenstern utility for the bundle $S \subseteq G$. We require $u(\emptyset) = 0$. We say that $u$ satisfies $k$-demand preferences if $u(\cdot)$ has one of three properties listed below. The first is that no agent has preferences for bundles that are too large, i.e.

$$u(S) = 0 \ \forall |S| \geq k + 1. \tag{1}$$

The second is a special case of the first that allows for free disposal and ensures monotonicity.

$$u(S) = \max_{A \subseteq S:} \{u(A) : |A| \leq k\} \ S \subseteq G. \tag{2}$$

The course allocation problem is a natural setting that gives rise to preferences that satisfy (1) or (2). Here students have preferences on bundles of objects that correspond to courses. Time constraints mean that each student can take only a limited number of course each quarter or semester. At Northwestern University, for example, $k$ is typically no more than 4 or 5 and $s_j$ a factor of 10 larger than that.

The third allows for agents to have preferences for large bundles. However, utility for a large bundle must be additive in 'smaller' sub-bundles. Specifically, assume a partition $P_1, \ldots, P_t$ of $G$ such that $|P_r| \leq k$ for all $r = 1, \ldots, t$[3]. Then,

$$u(S) = \sum_{r=1}^{t} u(S \cap P_r). \tag{3}$$

One instance of where preferences of the form (3) may arise is when the objects to be allocated are bands of spectrum. Bands of spectrum that interfere have similar frequency and are located close to one another and so can be categorized in groups of small size. Utilities for interfering bands need not be additive. Bands further apart do not interfere, so utilities for non-interfering bands can be taken to be additive.

(2) is a special case of (1). Neither imply or are implied by (3). When $k = 1$, (1-2) yield unit demand preferences and the combinatorial assignment problem reduces to the classical assignment problem. When $k = 1$, (3) means that utilities are additive.[4] Thus, $k$-demand preferences for $k = 1$ contain the basic classes of preferences that are substitutable. When $k = |G|$, $k$-demand preferences impose no restriction on preferences. Intermediate values of $k$ restrict the range of complementarities that can be expressed.

The IRM for allocating indivisible goods when agents have $k$-demand preferences produces a *randomized* mechanism that returns an allocation that maximizes total utility subject to

1. being ex-ante envy-free,
2. asymptotically strategy-proof,
3. and ex-post, the IRM will over-allocate each good by at most $k - 1$.

The last item on the list is of interest when $k$ is small relative to $s_j$ for each $j \in G$. In the course allocation problem, $k$ will be at most 4 while $s_j$ is usually at least 20 and frequently much larger. One can interpret this to mean that under $k$-demand preferences, by withholding a relatively small amount of each good, it will be possible to obtain allocations with desirable properties. Note that in the special case when $k = 1$ we recover the Birkhoff-von Neumann Theorem.

[3]Notice that we allow different agents to have different partition.

[4]For the motivation behind (3) see section 3.

## 1.1 Related Work

The most closely related paper to ours is Budish [2011]. Budish's mechanism is deterministic and based on computing an approximate competitive equilibrium from equal incomes. Thus, the preference information needed from agents is ordinal as opposed to cardinal in our case. Budish's mechanism returns an allocation that is approximately pareto-optimal, approximately envy-free in an ex-post sense, asymptotically strategy-proof and, like ours, violates the resource constraints. Budish bounds the violation in terms of the Euclidean distance ($O(\sqrt{\min\{2k, |G|\}|G|})$) between the supply vector and the vector of the number of goods allocated, unlike the bound on the maximum violation in each type of goods considered in this paper. In some settings the latter guarantee may be preferred to the aggregate error as in Budish [2011]. Lastly, as already mentioned earlier, unlike Budish [2011], our mechanism does not require the computation of a fixed point but the solution of a linear program, which can be solved in time polynomial in $|N|$ and $|G|$.

More recently, Peivandi [2013] introduced two other related mechanisms. These mechanisms, assume agents have ordinal preferences only, and have several attractive properties such as being ex-ante envy-free and ex-post feasible, and ordinarily efficient. However, the computational complexity of both of the mechanisms are exponential in the number of goods and agents.

The precursors to the papers just mentioned are the well known papers of Sha, Shapley and Scarf [1974], Demange et al. [1986] and Bogomolnaia and Moulin [2001]. These papers assume that no agent consumes more than one object (unit demand). In this setting any fractional assignment can be expressed as a convex combination of feasible integral assignments (a consequence of the Birkhoff-von Neuman Theorem). Thus, for example in the quasi-linear case, efficient allocations can be supported with Walrasian prices. In the non-transferable case, fractional assignments can be interpreted as lotteries over feasible integral allocations. When the unit demand assumption is relaxed, fractional assignments are not guaranteed to lie in the convex hull of feasible integral allocations.[5]

Our paper can be viewed as a generalization to the multi-unit demand setting using an approximate version of the Birkhoff-von Neuman Theorem that relaxes feasibility. In particular, when complementarities are limited, i.e., the maximum size of bundles that agents desire is small, the violation in feasibility is small.[6]

[5]Budish et al. [2013] uses standard results about the integrality of polyhedra to identify cases where fractional assignments can be interpreted as lotteries over feasible integral allocations.

[6]In many applications this potential infeasibility can be addressed by preemptively reducing supply. In other cases, like course assignment, substitutes for scarce resources can be summoned. For example, additional chairs are placed in the class.

In the next section of this paper we introduce notation and definitions. Section 3 formulates the an integer linear programming (LIP), that identifies a social welfare maximizing allocation of goods to agents subject to a standard 'fairness' constraint called envy-freeness. We then discuss how the solution of the linear relaxation of (LIP) is asymptotically strategy-proof. The subsequent section discusses how to implement the fractional solution of (LIP) as a lottery over approximately feasible integer solutions. Section 5 describes some applications of this technique to settings like school choice with side constraints and other type of resource allocation problem with transferable utility not previously discussed.

# 2 Notation and Model

Let $N$ be the set of agents. We assume each agent has one of a finite number of types $T$. For each type $t \in T$, let $n_t$ be the number of agents of type $t$. Let $G$ the set of goods and for each $j \in G$, let $s_j$ be the integer supply of good $j$. Thus, each instance of a market can be captured by a pair of vectors $(\vec{n}, \vec{s})$.

The type of an agent encodes their preferences which are represented by a von-Neuman Morgenstern **utility function** defined on bundles of goods. In our main applications (course allocation and school choice) we will assume that each agent wishes to consume at most one copy of each good.[7]

For an agent of type $t \in T$, let $u^t(S) \geq 0$ be his utility for bundle $S \subset G$. We will also use the notation $u_i^t(S)$ (or for short $u_i(S)$ ) for the utility of agent $i$ for bundle $S$ when his type is $t$. We assume that an agent's utility depends exclusively on his type and outcome. Furthermore, we assume for each type of agents, the utility function satisfies either (1), (2) or (3). Without loss of generality, we also assume $u^t(\emptyset) = 0$ for all type $t$. Given a **lottery** (a probability distribution) over a set of bundles, an agent's utility is his expected utility of the lottery.

A **feasible allocation** is an assignment of bundles of goods to agents such that the number of copies of each good $j$ allocated is at most the supply $s_j$.

An **$l$-approximately feasible allocation** is an assignment of bundles of goods to agents such that for each good $j$ the number of copies allocated to agents is at most $s_j + l$.

Given a type profile $\vec{t}$, an allocation $\vec{x}$ is **envy-free** if all agents weakly prefer the

[7]This assumption can be relaxed without influencing our results.

bundle assigned to them under $\vec{x}$ to any bundle assigned to another. That is,

$$u^{t_i}(x_i) \geq u^{t_i}(x_j)$$

Given a type profile $\vec{t}$, an allocation $\vec{x}$ is **efficient** (subject to envy-free) if among all the feasible allocations satisfying the envy-free constraints, $x$ optimizes

$$\sum_i u_i^{t_i}(x_i).$$

Let $A$ denote the set of (approximately) feasible allocations. For every $|N| > 0$, a mechanism $\Phi^{(N)}$ is a mapping from a profile of agents' types to a lottery over (approximately) feasible allocations. More precisely,

$$\Phi^{(N)} : T^N \to \Delta(A),$$

Without ambiguity, we sometimes use $\Phi$ instead of $\Phi^{(N)}$ for short.

A mechanism $\Phi$ is **polynomial time computable** if $\Phi$ is polynomial time computable given the input $u_t(S)$ for all $t \in T; S \subset G$; and the capacities $s_i$ for $i \in G$ as well as the report vector $\vec{t} \in T^N$.

It will be useful to consider a mechanism from the perspective of an agent $i$. Let

$$\Phi_i^{(N)} : T \times T^{N-1} \to \Delta(A_i),$$

where $A_i$ denote the possible bundles that agent $i$ obtains, and $\Phi_i(t_i, t_{-i})$ denotes the lottery over bundles that agent $i$ receives when he reports $t_i$ and other agents report $t_{-i}$.

A mechanism is $\Phi$ is **strategy-proof** if it is optimal for each agent to truthfully report their type given any vector of type reports of the other agents, that is

$$u^{t_i}(\Phi_i(t_i, t_{-i})) \geq u^{t_i}(\Phi_i(t'_i, t_{-i})).$$

A mechanism is **$\epsilon$- strategy-proof** if it is "almost'' optimal for each agent to report truthfully given any vector of reports by the other agents, that is

$$u^{t_i}(\Phi_i(t_i, t_{-i})) \geq u^{t_i}(\Phi_i(t'_i, t_{-i})) - \epsilon.$$

Finally, we define asymptotically strategy-proof. $\Phi$ is **asymptotically strategy-proof** if for any $\epsilon > 0$ there exists a constant $n_0$ such that $\Phi$ is an $\epsilon$-strategy-proof whenever there are at least $n_0$ agents reporting $r$ to $\Phi$ for each every type $r \in T$.

**Definition 2.1** *$\Phi$ is* **asymptotically strategy-proof** *if for any $\epsilon > 0$ there exists a constant $n_0$ such that if $\vec{t}$ satisfies $|\{i|t_i = r\}| \geq n_0$ for all $r \in T$, then*

$$u^{t_i}(\Phi_i(t_i, t_{-i})) \geq u^{t_i}(\Phi_i(t'_i, t_{-i})) - \epsilon.$$

Our definition of asymptotically strategy-proofness is similar in spirit to the notion of 'strategy-proofness in the large' introduced by Azevedo and Budish [2012]. To define this notion assume agents' reports are drawn *independently* from a distribution on the type set $T$ with full support. A mechanism is strategy-proof in the large if it is $\epsilon$ strategy-proof when the number of agents is large enough. In fact, any mechanism that is asymptotically strategy-proof in our sense will also be strategy-proof in the large.

# 3 Our Mechanism

It is well known that strategy-proofness, envy-freeness and efficiency are generally incompatible. Simply computing an efficient outcome in combinatorial settings is usually NP-hard. Here we construct a polynomial time computable mechanism that is asymptotically strategy-proof and subject to envy-freeness is "almost" efficient. In particular, we show that our mechanism achieves at least the optimal social welfare that any envy-free mechanism does while over-allocating each good by only a small amount.[8]

In this section we formulate a a linear program, and show that if a mechanism can implement the solution of this linear program, then, this mechanism is asymptotically strategy-proof, ex-ante envy-free and efficient. In the subsequent section we show that if each agents' utility satisfies (1), (2), or (3), then the solution of this linear program can always be implemented as a lottery over $k-1$-approximate feasible allocations. For ease of exposition we assume that utilities satisfy (1). The proof is the same for (2). For the the third class of utilities (3), the LP formulation and the proofs are provided in Appendix 6.2.

To describe the set of feasible allocations of objects to agents let $0 \leq x_i(S) \leq 1$ be the variable capturing the probability that agent $i$ obtains bundle $S$. Because of (1), each agent is only interested in bundles of size at most $k$, thus $x_i(S) = 0$ for all $S$ of size larger than $k$.

[8] One can enforce the hard constraints on the resources by scaling down the resources before applying our mechanism.

First, because each agent receives one bundle of goods (possibly empty), any allocation will need to satisfy the following constraint.

$$x \geq 0$$
$$\sum_{S \subseteq G; |S| \leq k} x_i(S) = 1 \; \forall i \in N \tag{DEMAND}$$

Feasibility requires that for each type of good $j$ we do not allocate more than its available supply:

$$\sum_{i \in N} \sum_{S \ni j} x_i(S) \leq s_j \; \forall j \in G \tag{SUPPLY}$$

The envy-free constraints of an allocation $x$ is the following.

$$\sum_{S \subseteq G} u_i^t(S) x_i^t(S) \geq \sum_{S \subseteq G} u_i^t(S) x_j^r(S) \; \forall i \; \forall j \neq i \; \forall r \neq t. \tag{ENVY-FREE}$$

Thus, any envy-free mechanism can achieve at most the welfare of the following program.

$$\max\{\sum_{i \in N} \sum_{j \in G} u_i(S) x_i(S) : \text{ s.t. } (DEMAND), (SUPPLY), (ENVY - FREE)\} \tag{LIP}$$

Call a fractional solution $x^*$ to (LIP) *implementable* if it can be expressed as a convex combination of feasible integer solutions to this program. An implementable fractional solution can be interpreted as a lottery over feasible integer solutions. Generally, $x^*$ is not implementable, but if it is, then clearly a mechanism that takes as input a report of each agents utility function and returns the optimal (fractional) solution to program (LIP) and implements it as a lottery is an envy-free and efficient mechanism. We will show in the next section how to implement $x^*$ as a lottery over approximately feasible allocations.

In the rest of this section we show that if (LIP) has a unique optimal solution the mechanism implementing $x^*$ is actually asymptotically strategy-proof. Notice that the assumption that (LIP) has an unique optimal solution is a mild assumption because we can always guarantee this by perturbing the objective function slightly .

**THEOREM 3.1** *Suppose (LIP) has an unique optimal solution $x^*$ which is implementable by a lottery $\bar{x}$, then the mechanism that takes in as input a report of each agents type and returns $\bar{x}$ is asymptotically strategy-proof.*

**Proof:** Recall that the set of types is a finite set $T$. Let $t_i$ be the type reported by agent

$i$, and let $n_t$ be the number of agents reporting type $t$.

Consider the following program for finding a utilitarian allocation that is envy-free:

$$\max \sum_{i \in N^*} \sum_{S \subseteq G} u_i^{t_i}(S) x_i^{t_i}(S) \tag{4}$$

$$\sum_{S \subseteq G} x_i^{t_i}(S) \leq 1 \ \forall i \in N \ \ \forall i \in N^* \tag{5}$$

$$\sum_{i \in N} \sum_{S \ni j} x_i^{t_i}(S) \leq s_j \ \forall j \in G \tag{6}$$

$$\sum_{S \subseteq G} u_i^{t_i}(S) x_i^{t_i}(S) \geq \sum_{S \subseteq G} u_i^{t_i}(S) x_j^{t_j}(S) \ \forall i \ \forall j \neq i \ \text{ and } t_i \neq t_j \tag{7}$$

Recall, that we can set $x_i^{t_i}(S) = 0$ where $|S| > k$. Call (4-7) the disaggregate formulation.

Introduce variables $y^t(S)$ to denote the 'aggregate' amount of bundle $S$ that all agents reporting $t$ get. Namely if we consider an anonymous solution of (4-7), that is $x_i^{t_i}(S) = x_j^{t_j}(S)$ whenever $t_i = t_j = t$, then $y^t(S) = n_t x_i^{t_i}(S)$. Now consider the following 'aggregate' formulation.

$$\max \sum_{t \in T} \sum_{S \subseteq G} u^t(S) y^t(S) \tag{8}$$

$$\sum_{S \subseteq G} \frac{1}{n_t} y^t(S) \leq 1 \ \forall t \in T \tag{9}$$

$$\sum_{t \in T} \sum_{S \ni j} y^t(S) \leq s_j \ \forall j \in G \tag{10}$$

$$\frac{1}{n_t} \sum_{S \subseteq G} u^t(S) y^t(S) \geq \frac{1}{n_r} \sum_{S \subseteq G} u^t(S) y^r(S) \ \forall t, r \in T, t \neq r. \tag{11}$$

To show that our mechanism is asymptotically strategy proof, we need to prove that for every $\epsilon > 0$, there exists $n_o$ such that if $n_t \geq n_0$ for all $t \in T$, then no agent can improve his utility by more than $\epsilon$.

Suppose agent $i$ of type $p$ pretends to be of type $q$. We will show that the impact on the allocations of the other agents from this misreport can be computed by solving (8-11) with a perturbed right hand side.

If agent $i$ of type $p$ pretends to be of type $q$ then the number of agents reposting $p$ is decreased by one and the number of agents reposting $q$ is increased by 1. Thus, the aggregate formulation becomes:

$$\max \frac{n_p - 1}{n_p} \sum_{S \subseteq G} u^p(S) y^p(S) + \frac{n_q + 1}{n_q} \sum_{S \subseteq G} u^q(S) y^q(S) + \sum_{t \in T - \{p,q\}} \sum_{S \subseteq G} u^t(S) y^t(S) \tag{12}$$

$$\sum_{S \subseteq G} y^t(S) \leq 1 \ \forall t \in T \tag{13}$$

$$\frac{n_p - 1}{n_p} \sum_{S \ni j} y^p(S) + \frac{n_q + 1}{n_q} \sum_{S \ni j} y^q(S) + \sum_{t \in T - \{p,q\}} \sum_{S \ni j} y^t(S) \leq s_j \ \forall j \in G \tag{14}$$

$$\frac{1}{n_t} \sum_{S \subseteq G} u^t(S) y^t(S) \geq \frac{1}{n_r} \sum_{S \subseteq G} u^t(S) y^r(S) \ \forall t, r \in T, t \neq r. \tag{15}$$

Compare program (12)-(15) to program (8)-(11). If both $n_p$ and $n_q$ are large enough then the objective function and the constraints of both program are close to each other. Because of the assumption that (8)-(11) has an unique solution, thus, there must $n_0$ such that if both $n_p$ and $n_q$ are at least $n_0$ then (12)-(15) also has an unique solution. Furthermore, as $n_0$ in creases the solution of (12)-(15) is converging to the solution of (8)-(11).

In other words, if $n_0$ is large enough then the agent who misreports their type can only change their allocation by $O(\epsilon)$. Thus, by the envy-free constraint, their utility changes by at most $O(\epsilon)$. This shows that the mechanism is asymptotically strategy proof. ■

Under the implementability assumption, Theorem 3 yields a mechanism that is asymptotically strategy proof, ex-ante envy free and among all such mechanisms maximizes total welfare. As we noted earlier, the implementability assumption is usually false. However, in the next section we show that under the $k$-demand preference assumption every fractional solution to $(DEMAND), (SUPPLY), (ENVY-FREE)$ is 'approximately' implementable in a well defined sense.

# 4 Approximate Implemention

In this section we show that fractional solutions to $(DEMAND)$ and $(SUPPLY)$ are approximately implementable. Exact implementation is known only in certain specials cases (Birkhoff-von Neuman, Budish et al. [2013]). These exact results are obtained by showing that the polytope of feasible allocations is integral. Hence, any fractional solution can be decomposed into a convex combination of integral allocations. This property fails in our setting as the polytope defined by $(DEMAND)$ and $(SUPPLY)$ is not integral. Nevertheless, as we show below, we can decompose any fractional solution into a lottery over approximately feasible integer allocations. The first step in the argument is the use of the Iterative Rounding Method (IRM) of Király et al. [2008] which we now describe.

Let $P$ be the polytope defined by $(DEMAND)$ and $(SUPPLY)$. The IRM takes as input an extreme point, $x^* \in \arg\max\{u \cdot x : x \in P\}$ where $u \geq 0$ and $u_i(S) = 0$ for all $i \in N$ and $S \subseteq G$ such that $|S| > k$. It then rounds $x^*$ into a 0-1 vector $\bar{x}$ that satisfies (DEMAND) and is such that

$$\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1 \; \forall j \in G. \qquad \text{(SUPPLY-k)}$$

Beginning with $x^*$, we remove from (DEMAND-SUPPLY) all variables $x_i(S)$ for which $x^*_i(S) = 0$. In other words, a variable that is zero in $x^*$ will be rounded down to zero and fixed at that value in all subsequent iterations. Similarly, remove from (DEMAND-SUPPLY) all variables $x_i(S)$ for which $x^*_i(S) = 1$ and adjust the right hand sides of (SUPPLY) accordingly. In other words, a variable set to 1 by $x^*$ is fixed at 1 in all subsequent iterations. In the system that remains pick a non-negative extreme point that optimizes the vector $u$ and repeat. At some iteration, when the remaining supply of good $j$ is $s'_j$, we may obtain an extreme point with no variable set to 1. Call it $y$.

The main observation here is that, in this case there must exist a $j \in G$ such that

$$\sum_{i \in N} \lceil \sum_{S \ni j} y^*_i(S) \rceil \leq s'_j + k - 1.$$

For each such $j$, remove the corresponding constraint (SUPPLY) and in the relaxed system find an extreme point that optimizes $u$ and repeat. Stop once all variables have been fixed at either 0 or 1 and denote the resulting 0-1 vector by $\bar{x}$.

There are three observations to be made about $\bar{x}$.

1. At each iteration, inequality (DEMAND) holds. Thus, $\bar{x}$ satisfies (DEMAND).

2. At each iteration, the original program is (possibly) relaxed. Thus, $u \cdot \bar{x} \geq u \cdot x^*$.

3. Because $\bar{x}_i(S) = 1$ only if $x^*_i(S) > 0$, it follows that for the inequalities in (DEMAND) thrown away, $\sum_{i \in N} \sum_{S \ni j} \bar{x}_i(S) \leq s_j + k - 1$.

The key lemma is the following:

**Lemma 4.1** *Let $u_i(S) \geq 0$, $u_i(\emptyset) = 0$ and $u_i(S) = 0$ for all $|S| > k$ and $i \in N$. Let $x^*$ be an extreme point of $P$ in $\arg\max\{u \cdot x : x \in P\}$ such that $x_i^*(S) < 1$ for all $i \in N$ and $S \subseteq G$. Then, there exists a $j \in G$ such that*

$$|\{i \in N : \sum_{S \ni j} x_i^*(S) > 0\}| \leq s_j + k - 1.$$

$$\sum_{i \in N} \lceil \sum_{S \ni j} x_i^*(S) \rceil \leq s_j + k - 1.$$

For completeness a proof is given in Appendix 6.1.

This rounding protocol gives the following result proved by Király, Lau and Singh (2008), which is restated as follow.

COROLLARY 4.2 *Given any (not necessarily non-negative) utility vector $u_i(S)$ and any fractional vector $x$ satisfying (DEMAND) and (SUPPLY), we can find in polynomial time an* integral *vector $x^*$ satisfying (DEMAND) and (SUPPLY-k) such that $ux^* \geq ux$.*

With this, we are ready to state the main result.

**THEOREM 4.3** *Any $x$ satisfying (DEMAND) and (SUPPLY) can be expressed as a convex combination of integral vectors satisfying (DEMAND) and (SUPPLY-k). Furthermore, in polynomial time we can find a convex combination that is arbitrarily close to $x$.*

**Proof:** To the easy of presentation, let $Q$ be the polytope consisting of all *real* vectors satisfying (DEMAND) and (SUPPLY); and let $E_k$ be the set of *integral* solutions to (DEMAND) and (SUPPLY-k).

Suppose theorem 4.3 does not hold. Thus, there is a $x \in Q$ that is not in the convex hull of $E_k$. Hence, there must be a hyperplane that separates $x$ from $E_k$. Let $u$ be the vector of coefficients of that hyperplane. Choose it so that $ux > uz$ for all $z \in E_k$. But this is a contradiction to Corollary 4.2.

The proof above is nonconstructive. In Appendix 6.3, we provide a polynomial time algorithm using IRM repeatedly to compute a lottery over integral solutions of $E_k$ whose expectation is arbitrarily close to the given vector in $Q$.

∎

# 5 Other Applications

## 5.1 School Choice

In this section we consider an environment that satisfies the unit demand constraint but must satisfy additional constraints. Such problems are motivated by school assignment problems where students must be assigned to schools in a way to ensure diversity on various dimensions (see Ehlers et al. [2011] for example). Many school districts such as Chicago, Boston and New York are concerned with issues of student diversity and have included affirmative action systems into their school choice program.

In this setting we can consider each school as a type of goods. Let $s_j$ be the number of capacity of school $j$. As before let $G$ be a set of goods and $N$ a set of agents (students). However in this case each agent wishes to consume at most one good. Let $x_{ij} = 1$ if agent $i \in N$ is assigned good $j \in G$ and zero otherwise. To model the diversity constants, we assume each agent is also endowed with a 0-1 vector of dimension $k$ that records which of $k$ binary characteristics she possesses. Examples of characteristics are gender, race or citizenship. Denote by $C^r$ the set of agents with characteristic $r = 1, \ldots, k$. An assignment of goods to agents is feasible if it satisfies the following

$$\sum_{j \in G} x_{ij} = 1 \ \forall i \in N \tag{16}$$

$$\sum_{i \in N} x_{ij} \le s_j \ \forall j \in G \tag{17}$$

$$q_j^r \le \sum_{i \in C^r} x_{ij} \le Q_j^r \ \forall r = 1, \ldots, k, \ \forall j \in G \tag{18}$$

The envy-free constraint is the following.

$$\sum_j u_i(j) x_{ij} \ge \sum_j u_i(j) x_{i'j} \quad \forall i' \ne i \tag{19}$$

We assume that (16- 19) is non-empty. As in the combinatorial assignment problem, we consider the following linear program

$$\max\{\sum_i u_i(j) x_{ij} : \quad \text{s.t } (16 - 19)\} \tag{20}$$

Our goal is to implement an allocation achieving a welfare as good as the solution of (20).

However, we will need to scarify on feasibility. Our mechanism will output allocations that satisfy a weaker feasibility constants than (17- 18). Namely,

$$\sum_{i\in N} x_{ij} \le s_j + 2k + 1 \;\forall j \in G \tag{21}$$

$$q_j^r - 2k - 1 \le \sum_{i\in C^r} x_{ij} \le Q_j^r + 2k + 1 \;\forall r = 1, \dots, k,\; \forall j \in G, \tag{22}$$

where $k$ is the maximum number characteristics that a student possesses.

**THEOREM 5.1** *There exists a polynomial time computable mechanism that is envy-free; asymptotically strategy proof and achieves the optimal welfare of (20). Furthermore, the mechanism outputs approximately feasible allocations that satisfy (21-22).*

**Proof:** The approach and the mechanism for the school choice problem above is similar to the combinatorial assignment problem discussed in the previous section. We first solve (20), and then using the rounding protocol to decompose a solution that satisfies (21) and (22) into a lottery over allocations that satisfy (21-22). The argument uses the iterative rounding method in Király et al. [2008], which shows that for any fractional solution $x$ satisfying (16- 18), and any cost function $c$, we can find an integral $x^*$ satisfying (16, 21 , 22) and $c^*x \ge cx$. Using this result, the proof of our theorem is analogous that of Theorem 4.3. ■

## 5.2 Walrasian Prices

While the main application of our technique was to the allocation of indivisible goods when transfers were not permitted, it is not limited to these cases. We illustrate with an application to settings where transfers are allowed.

Suppose agents have quasi-linear preferences and let $u_i(S)$ be the monetary value that agent $i \in N$ assigns to bundle $S \subseteq G$. In this setting it is well known that without further restrictions on the preferences of the agents, Walrasian prices that clear the market need not exist.

**THEOREM 5.2** *If all agents have $k-$ demand preferences, that is agents' utility satisfies (1), (2), or (3), there exist Walrasian prices where the excess demand for any good is at most $k-1$.*

**Proof:** We give the proof for the case when preferences satisfy (1). A similar proof applies in the case (2) and (3). To find an efficient allocation we solve $\max\{u \cdot x : x \in P\}$. In this program set $u_i(S) = 0$ for all $S$ such that $|S| > k$. This does not change the value of the efficient allocation but does shrink the set of efficient allocations.

Let $p^*$ be the optimal dual variables associated with (DEMAND) in the program $\max\{u \cdot x : x \in P\}$. We can interpret $p^*$ as a Walrasian price vector. For each $i \in N$ let $D_i(p^*) = \arg\max_{S \subseteq G}[u_i(S) - \sum_{j \in S} p_j^*]$, i.e., agent $i$'s demand correspondence. Lemma 4.1 implies that we can give to each agent an element of their demand correspondence at price $p^*$ such that (SUPPLY-k) is satisfied. ■

Informally, as long as agents do not demand bundles that are very large, or the complementaries in their preferences are limited, there exist Walrasian prices that approximately clear the market.

## 5.3 Truthful Combinatorial Auctions

In our second example, consider the problem of designing an individually rational, weakly dominant strategy, efficient mechanism in this setting. It is well known that the Vickrey-Clarke-Groves mechanism will do the trick. However, computing the efficient allocation is NP-hard.[9]

**THEOREM 5.3** *Suppose all agents have preferences that satisfy (1), (2), or (3). The IRM yields a polynomial time mechanism that is individually rational, weakly dominant strategy, over allocates at most $k-1$ copies of each good and generates an expected total welfare at least as large as the efficient allocation.*

**Proof:** To see this, let $x^* \in \arg\max\{u \cdot x : x \in P\}$. As before we assume that $u_i(S) = 0$ whenever $|S| > k$. Observe that $x_i^*(T^i)$ is monotone in $u_i(T^i)$. By Theorem 4.3 we can represent $x^*$ as a lottery over $E_k$. Thus, in expectation, the welfare of the allocation rule defined by this lottery is at least as large as the welfare of the efficient allocation. Hence, this lottery can be implemented in a way that is incentive compatible in expectation (the expectation is with respect to the randomization induced by the lottery). Furthermore, in the worst case, our mechanism over allocates at most $k-1$ copies of each good. ■

Thus, if each good has a large number of copies, this mechanism will yield an almost feasible allocation.

[9] In the last decade there has been an extensive literature on designing *polynomial time* mechanisms for combinatorial auctions. See for example Blumrosen and Nisan for a survey.

# 6 Conclusion

In this paper developed a general method for designing a mechanism to allocate indivisible goods to agents without the use of transfer. This framework allows for more general settings than the traditional unit demand assumption. We show that when agents do not demand too large bundles, one can obtain allocations with a number of attractive features (social welfare maximizing; fairness; polynomial computable) by discarding a relatively small number of the goods.

Our mechanism applies to a broad range of applications. We hope to see more of its applications in other setting, as well as learn more about the empirical performance of this approach.

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## References

A. Abdulkadiroglu and T. Sönmez. School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747, June 2003.

E. M. Azevedo and E. Budish. Strategyproofness in the large as a desideratum for market design. In *ACM Conference on Electronic Commerce*, page 55, 2012.

L. Blumrosen and N. Nisan. *Combinatorial Auctions (a survey) In Algorithmic Game Theory , by N. Nisan, T. Roughgarden, E. Tardos and V. Vazirani, editors.*

A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, October 2001.

E. Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061 – 1103, 2011.

E. Budish. Matching versus mechanism design. *ACM SIGecom Exchanges*, 11(2):4–15, 2012.

E. Budish, Y.-K. Che, F. Kojima, and P. Milgrom. Designing random allocation mechanisms: Theory and applications. *American Economic Review*, 103(2):585–623, September 2013.

G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–72, August 1986.

L. Ehlers, I. Hafalir, B. Yenmez, and M. Yildirim. School choice with controlled choice constraints: Hard bounds versus soft bounds. GSIA Working Papers 2012-E20, Carnegie Mellon University, Tepper School of Business, Nov. 2011.

A. Hylland and R. Zeckhauser. The efficient allocation of individuals to positions. *Journal of Political Economy*, 87(2):293–314, April 1979.

T. Király, L. C. Lau, and M. Singh. Degree bounded matroids and submodular flows. In *Proceedings of the 13th international conference on Integer programming and combinatorial optimization*, IPCO'08, pages 259–272, Berlin, Heidelberg, 2008. Springer-Verlag.

L.-C. Lau, R. Ravi, and M. Singh. *Iterative Methods in Combinatorial Optimization*. Cambridge University Press, New York, NY, USA, 1st edition, 2011. ISBN 0521189438, 9780521189439.

A. Peivandi. Random allocation of bundles. *Working paper, Northwestern University*, 2013.

L. Shapley and H. Scarf. On cores and indivisibility. *Journal of Mathematical Economics*, 1(1):23 – 37, 1974. ISSN 0304-4068.

# Appendix

## 6.1 Proof of Lemma 4.1

We will use the following property of an extreme point of a linear program:

> The number of non-zero variables in an extreme point $x^*$ is equal to the number of linearly independent and binding constraints in (DEMAND) and (SUPPLY).

To prove the lemma, assume that for all $j \in G$

$$|\{i \in N : x_i^*(S) > 0,\ S \ni j\}| > s_j + k, \tag{23}$$

we will derive a contradiction to the property of the extreme point above.

Given the extreme point $x^*$, where we credit each non-zero variable $x_i^*(S)$ with a single token. We then redistribute these tokens to the binding, linearly independent constraints in a particular way. We show that if (23) holds then each binding constraint will get at least one token, and there is one token left over. This shows that the number of non-zero variable $x_i^*(S)$ is larger than the number of binding, linearly independent constraints, which is a contradiction.

We redistribute the tokens given as follows. Credit $x_i^*(S)$ fraction of the tokens to the constraint corresponding to agent $i$ (DEMAND). Credit $\frac{1-x_i^*(S)}{k}$ to each constraint corresponding to each good $j \in S$. Notice that this is feasible because there are at most $k$ such constraints.

If the constraint corresponding to agent $i$ binds then the number of tokens this constraint is credited with is $\sum_S x_i(S) = 1$. Now, consider a binding constraint corresponding to good $j$, we have.

$$\sum_{i \in N} \sum_{S \ni j} x_i(S) = s_j.$$

Let $K = |\{i \in N : x_i^*(S) > 0,\ S \ni j\}|$, that is the number of non-zero variable that contribute to the constraint for the good $j$. Because each of these variable contribute to the constraint $\frac{1-x_i^*(S)}{k}$, therefore, overall the constraint obtains

$$\frac{K - \sum_{i \in N} \sum_{S \ni j} x_i(S)}{k} = \frac{K - s_j}{k}$$

tokens. Because $K \geq s_j + k$ by (23), this constraint is credited with at least 1 token.

Thus, we have shown that the amount of tokens given at the beginning (which is the number of non-zero $x^*$ variables) has been redistributed to the binding constraints, so that each is credited with at least 1 token. Thus the number of non-zero $x^*$ variables is at least the number of binding constraints. Now, equality obtains only if for every nonzero $x_i^*(S)$, $|S| = k$, and the constraint corresponding to agent $i$ binds, furthermore, for all $j \in S$ the constraint corresponding to $j$ also binds.

But in this case one can show that the set of binding constraints is not linearly independent. To see this, consider the sum of all the binding constraints in SUPPLY:

$$\sum_{j \in G} \sum_{i, S: S \ni j} x_i^*(S) = \sum_j s_j.$$

Because for each $x_i^*(S) > 0$, $|S| = k$, this sum can be rewritten as

$$k \cdot \sum_i \sum_S x_i^*(S) = \sum_j s_j.$$

This last expression is the sum of all the constraints in (DEMAND), contradicting linear independence of the binding constraints. By this we have shown that the number of nonzero variables in an extreme point solution is larger than the number of linearly independent binding constraints. ■

## 6.2 LP formulation for the class of utilities satisfying (3)

Recall that the third class of utilities that we consider allows for agents to have preferences for large bundles. However, utility for a large bundle must be additive in 'smaller' sub-bundles. Specifically, assume for an agent $i$, there is a partition $P_1^i, \ldots, P_{t_i}^i$ of $G$ such that $|P_r^i| \leq k$ for all $r = 1, \ldots, t_i$. Then,

$$u_i(S) = \sum_{r=1}^{t_i} u_i(S \cap P_r^i). \tag{24}$$

To formulate an LP for this class of utilities, here, instead of having the variable $x_i(S)$ for all $S \subset G$, we consider $x_i(S_r)$ for $r \in [1, .., t_i]$ and $S_r \subset P_r^i$. That is $x_i(S_r)$ indicates whether the sub-bundle $S_r$ among the goods in $P_r^i$ is allocated to agent $i$. We introduce the following Demand' constraints.

$$x \geq 0$$

$$\sum_{S_r \subset P_r^i} x_i(S_r) = 1 \ \forall i \in N, r \in [1, .., t_i] \qquad \text{(DEMAND')}$$

This is the only difference in the LP formulation for this class of utilities. Under the assumption (24), the new (DEMAND') constraints do not influence the linear program that maximizes the total social welfare.

It is also clear that our results on asymptotically strategy-proofness holds, thus it remains to see that the iterative rounding technique also works for this new class of constraints. In fact, (DEMAND') is a special case of matroid constraints (partition matroid), and as shown in Király et al. [2008] the IRM gives the same error bound guarantee for any matroid constraint. ■

## 6.3 Decomposition Algorithm

Recall that Theorem 4.3 shows that any $x \in Q$ can be expressed as a convex combination of points in $E_k$. In this section we show how to (approximately) decompose any $x \in Q$ into a convex combination of points in $E_k$.

Assume $E_k$ is bounded with diameter $D$. Denote by $|x - y|$ the Euclidean distance between $x$ and $y$. Recall that we have a subroutine that will for any fractional $x \in Q$ and any cost vector $c$, return an integral $\bar{x} \in E_k$ such that $c\bar{x} \geq cx$.

Given this subroutine, we exhibit a polynomial time algorithm that for a given point $x \in Q$, finds at most $d+1$ integral points in $E_k$ whose convex hull is arbitrarily close to $x$. The algorithm also returns a lottery over these $d + 1$ integral vectors whose expectation is close to $x$.

Given a fractional solution $x \in Q$. Let

$$B(x, \delta) = \{z : \text{ satisfying } (DEMAND) \text{ and } |z - x| \leq \delta\}$$

We assume there exists $\delta > 0$ such that $B(x, \delta) \in Q$. Notice that for our purpose, this assumption is without loss of generality, because otherwise we can always choose $x'$ in the interior of $Q$ close to $x$.

Given an allowable error $\epsilon > 0$, the algorithm is the following.

**Algorithm** In each step maintain a subset $S$ of points in $E_k$. Each iteration consists of the following steps.

1. Compute $y \in conv(S)$ that is closest to $x$. If $|y - x| < \epsilon$, the algorithm terminates.

2. Otherwise, because $y$ is the closest point to $x$ in $S$, $y$ lies in a hyperplane of $conv(S)$. Thus, there exists a subset $S' \subset S$ of size at most $d$ such that $y \in conv(S')$. (Recall $d$ is the dimension).

Consider $z = x + \delta \frac{x-y}{|x-y|}$. Notice, $z \in Q$ because $B(x, \delta) \in Q$. Use IRM to find an integral $z' \in E_k$, such that

$$< z, x - y > \leq < z', x - y > .$$

3. Update $S := S' \cup \{z'\}$; and repeat.

To show that the algorithm terminates in polynomial time, we show that after each iteration, the distance $|x - y|$ is reduced by at least a constant factor. To prove this, let $y'$ be the point in the interval $[z', y]$ that is closest to $x$. We will prove the following.

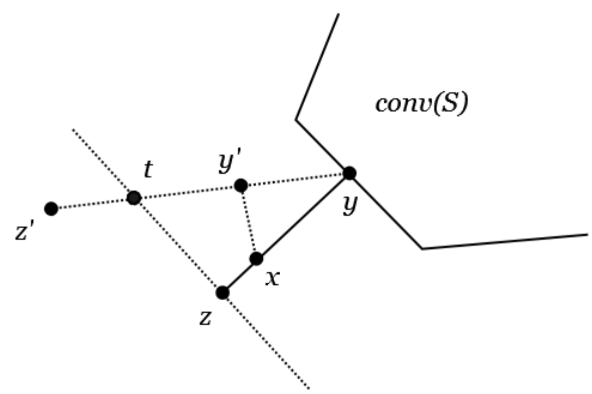


Figure 1:

CLAIM 6.1 *There exists* $0 < \gamma < 1$ *that depends on* $D$ *and* $\delta$ *such that* $|x - y'| < (1 - \gamma)|x - y|$

**Proof:** Let $t$ be the point in the interval $(z', y)$ such that $< t - z, x - y >= 0$. Because $< z, x - y > \leq < z', x - y >$, such a $t$ exists. See Figure 1.

Now,

$$\frac{|x-y|^2}{|x-y'|^2} = \frac{|t-y|^2}{|t-z|^2} = \frac{|t-z|^2 + (|x-z| + |x-y|)^2}{|t-z|^2} \geq \frac{|t-z|^2 + \delta^2}{|t-z|^2}$$

We have $|t - z| \leq |z' - z|$. Furthermore, because the diameter of $E_k$ is $D$, $|z' - z| \leq D$. Thus, $|t - z| \leq D$.

Hence, we obtain

$$\frac{|x-y|^2}{|x-y'|^2} \geq \frac{D^2 + \delta^2}{D^2}$$

Thus, there exists $0 < \gamma < 1$, depending on $D$ and $\delta$ such that

$$|x - y'| < (1 - \gamma)|x - y|,$$

which is what we need to prove. ■

The claim above shows that after each iteration the distance between $x$ and $y$ is reduced by at least a factor of $(1 - \gamma)$. Consider $K = \frac{\ln(D/\epsilon)}{\gamma}$, we have

$$D(1 - \gamma)^K \leq \epsilon,$$

Thus, after at most $K$ iterations, the algorithm will terminate.