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Coherently amplifying random medium: Statistics of super-reflection

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Light-wave propagation in a randomly scattering but uniformly and coherently amplifying optical medium is analysed for the statistics of the coefficient of reflection $r(l)$ that may now exceed unity, hence super-reflection. Uniform coherent amplification is introduced phenomenologically through a constant negative imaginary part added to the otherwise real dielectric constant of the medium, assumed random. The probability density $p(r, l)$ for the reflection coefficient, calculated in a random phase approximation, tends to an asymptotic form $p(r, \infty)$ having a long tail with divergent mean $\langle r \rangle$ in the limit of weak disorder but with the sample length l (measured in units of the localization length in the absence of gain) $\gg 1$. This super-reflection is attributed to a synergetic effect of localization and coherent amplification, as distinct from the classically diffusive path-length prolongation. Our treatment is based on the invariant-imbedding method for the one-channel (single-mode) case, and is in the spirit of the scattering approach to wave transport as pioneered by Landauer. Generalization to the multichannel case is pointed out. Relevance to random lasers, and some recent results for a sub-meanfree path sample are also briefly discussed.

Key words: random laser, localization, amplification, fluctuations.

1. Introduction

It is now well known that for the coherent wave transport through a mesoscopic, possibly disordered sample the notion of the usual ensemble-averaged transport coefficient such as the resistivity/conductivity, that is local and material specific, may have to be replaced by that of the resistance/conductance that is global and operationally specific to the sample as well as to the probe for measurement. The latter is best expressed in terms of an emergent quantity such as the reflection/transmission coefficient for the whole sample regarded as a single scatterer having an S-matrix. This remarkably simple and direct but powerful scattering approach to wave (quantum) transport was pioneered by Landauer [1] in an influential paper published almost a quarter of a century ago. It has now become central to the fast-developing field of what has come to be known as the mesoscopic physics. This approach readily generalizes from the case of the passive scattering medium to

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that of a medium which scatters actively—to the problem of a light-wave propagating through a random but coherently amplifying medium. This is the problem of random lasers which is, of course, the subject matter of this presentation [2]. The authors feel greatly privileged to have been asked to contribute to this special issue to mark the seventieth birthday of Professor Rolf Landauer.

Light-wave propagation through a spatially random but laser-active (gainful) dielectric medium is an excellent laboratory for studying the interplay of disorder-induced (Anderson) localization and the coherent amplification. (This is clearly not directly possible in the case of the electron wave because of its Fermionic nature). Essential to localization, however, is the coherent back-scattering (CBS) subtended by the quenched disorder, but the coherent amplification (by stimulated emission) happens to maintain this CBS condition—in fact it reinforces it by enhancing the contribution of the longer virtual return paths. Localization in turn acts as a perfect high-quality (virtual) cavity providing, thus, the necessary feedback which is quasiresonant. This synergy between the localization (due to quenched disorder) and the coherent amplification (due to stimulated emission) is theoretically expected to give a mirrorless random lasing [2–15], perhaps a broad-band one. It also poses an interesting question of the possible mode-selection by gain narrowing when the confinement is due entirely to quenched randomness.

Such an active (coherently amplifying) scattering medium is realizable, and has indeed been realized [16] as colloidal suspension of dielectric microspheres (e.g., rutile, TiO_2) in a fluid such as methanol giving high-contrast refractive index for strong scattering, and containing a lasing dye (e.g., rhodamine 640 perchlorate) which is then pumped externally. Coherent laser emission from such a random amplifying medium has been reported, and has become a subject of active research [2–20]. Recently, we had initiated a theoretical study [2] of light-wave propagation in a one-dimensional coherently amplifying medium where amplification is introduced phenomenologically by adding a negative imaginary part to the otherwise real dielectric constant, assumed spatially random. (Coherent amplification/attenuation can also be introduced formally through the introduction of *fake* side-channels [3]). In particular, we studied the statistics of the reflection coefficient $r(L)$ as a function of the disorder strength, medium gain and the sample length L. As our main result, we obtained super-reflection ($r > 1$) giving divergent $\langle r \rangle$ with probability 1 for L much greater than the localization length, as a clear evidence for the synergy between localization and coherent amplification. This is different from the idea of prolongation of the optical path length through the gain medium, namely that for a classical diffusion the arc length traversed scales as the square of the chord (L) [21]. In the following we describe our work briefly.

Consider the propagation of a monochromatic light wave described by the Maxwell equation for the electric vector

$$-\nabla^2 \boldsymbol{E} + \nabla(\nabla \cdot \boldsymbol{E}) - \frac{w^2}{c^2}\epsilon_r(x)\boldsymbol{E} = \epsilon_0 \left(\frac{w^2}{c^2}\right)\boldsymbol{E} \tag{1}$$

with w = circular frequency and c the speed of light *in vacuo*. The medium dielectric constant $\epsilon(x) = \epsilon_0 + \epsilon_r(x)$, where $\epsilon_r)x)$ is the random real part causing scattering. Further, we set $\epsilon_0 = \epsilon_0' - i\epsilon_0''$ with $\epsilon_0'' > 0$ so as to introduce coherent amplification in the medium.

But for the depolarization second term on the left-hand side of eqn (1), we have the Helmholtz equation well known for the localization problem. Indeed, for the unitary case (i.e., with $\epsilon_0'' = 0$), we have the random term $\epsilon_r(x)$ giving the usual Rayleigh scattering ($\propto \lambda^{-(1+D)}$) that suppresses localization in the long wavelengths $\lambda(\equiv 2\pi/k)$ limit in three dimensions (i.e., for $D = 3$), while the short wavelength limit corresponds to the classical geometrical optics with no localization anyway [22]. This is what makes it hard to satisfy the Mott–Ioffe–Regel localization criterion, $kl_s <\sim 2\pi$ in the three-dimensional case [22]. Here we consider the $D = 1$ case appropriate to the light-wave propagation in a single-mode, polarization-maintaining optical fibre disordered longitudinally. The depolarization term on the left-hand side of eqn (1) then drops out. Amplification can be realized in practice by, for example, Er^{3+} doping of the fibre and optical pumping. The $D = 1$ case is, of course, most favourable for light localization.

The resulting one-dimensional Helmholtz equation for the linearly polarized transverse field is

$$\frac{\partial^2 E}{\partial x^2} + k^2((1+\eta_r(x)) + i\eta_a)E = 0 \tag{2}$$

with $k^2 = (w^2/c^2)\epsilon_0'$, $\eta_r(x) = \epsilon_r(x)/\epsilon_0'$, and the amplification parameter $\eta_a = -\epsilon_0''/\epsilon_0'$.

In order to look for super-reflection and its statistics, we transform eqn (2) to the invariant imbedding equation [23, 24] for the complex amplitude reflection coefficient $R(L) = \sqrt{r(L)}\exp(i\theta(L))$ as

$$\frac{dR}{dL} = 2ikR(L) + \frac{ik}{2}(1+\eta_r(L)+i\eta_a)(1+R(L))^2 \tag{3}$$

with the *initial* condition $r(L=0)=0$. Equation (3) describes a random walk in the complex R-space with *time* L. Analytical results could be obtained for the case of a Gaussian white-noise disorder:

$$\langle\eta_r(L)\rangle = 0, \qquad \langle\eta_r(L)\eta_r(L')\rangle = \eta_0^2\delta(L-L'). \tag{4}$$

The Fokker–Planck equation for the marginal probability density $p(r,l)$ of the reflection coefficient associated with the stochastic differential eqn (3) then turns out to be

$$\frac{\partial_p(r,l)}{\partial l} = r(1-r)^2\frac{\partial^2 p(r,l)}{\partial r^2} + [1+(-6-D_s)r+5r^2]\cdot\frac{\partial p(r,l)}{\partial r} + [(-2+D_s)+4r]r(r,l), \tag{5}$$

where we have assumed a random phase approximation, i.e., statistical independence of r and θ and a uniform distribution for θ. We expect this approximation to be valid in the limit of weak disorder, weak amplification and long sample length $l \equiv L/l_s \gg 1$, where $l_s = 2/\eta_0^2k^2 = \xi \equiv$ the scattering (localization) length for one-dimension without gain. Also, $D_s = 4\eta_a/\eta_0^2k = l_s/l_a$ with $l_a = 1/2\eta_a k \equiv l_g$, the amplification (gain) length.

Equation (5) can be readily solved for the limiting case $l \gg 1$ when the distribution saturates, in fact quickly, to the asymptotic form. Accordingly setting $dp/dl = 0$ we get (dropping the subscript s from D_s):

$$\begin{aligned} p(r,\infty) &= \tfrac{-D}{(1-r)^2}\exp(D/(r-1)) && \text{for } r>1,\\ &= 0 && \text{for } r<1, \end{aligned} \tag{6}$$

for $D<0$ (coherent amplification).

The case of a coherent (stochastic) attenuation can also be discussed by simply changing the sign of D and normalizing:

$$\begin{aligned} p(r,\infty) &= \tfrac{D\exp D}{(1-r)^2}\exp(-D/(1-r)) && \text{for } r<1,\\ &= 0 && \text{for } r>1, \end{aligned} \tag{7}$$

for $D>0$ (for coherent attenuation).

It is readily verified from eqn (6) that for $D<0$ (amplification), we get super-reflection with probability unity in the long-length limit $l \gg 1$, with $\langle r\rangle$ divergent. Figure 1 plots $p(r,\infty)(\equiv p(r))$ from eqns (6) and (7). Figure 2 displays the evolution of $p(r)$ with the sample length l. These solutions have been numerically verified by Zhang [4].

Some remarks are in order now. First, note that the imbedding eqn (3) for $R(L)$ obeys a duality relation—it is invariant under $R(L) \rightleftharpoons 1/R(L)$ with $\eta_a \rightleftharpoons -\eta_a$. Thus, in terms of the probability distribution of $\ln R(L) \equiv \frac{1}{2}\ln r + i\theta(L)$, it equates the localization lengths for the two cases—amplification and attentuation. Similar conclusion has also been reached by other workers [6]. Equation (5) has been generalized to the N-channel case by Beenakker *et al.* [5] by following the Dorokhov-Mello–Pereyra–Kumar (DMPK) equation [25, 26], and the coherent nature of the super-reflected *albedo* thus obtained has been confirmed [5]. Now a technical point: in the imbedding method we consider the reflection from the sample attached to perfect leads—with $\epsilon_r = 0 = \epsilon_0''$). Thus, even in the absence of randomless ($e_r(x)=0$), the uniform imaginary part $(-iE_0'')$ assumed nonzero for the sample, does produce reflection due to the terminal mismatch. Inasmuch

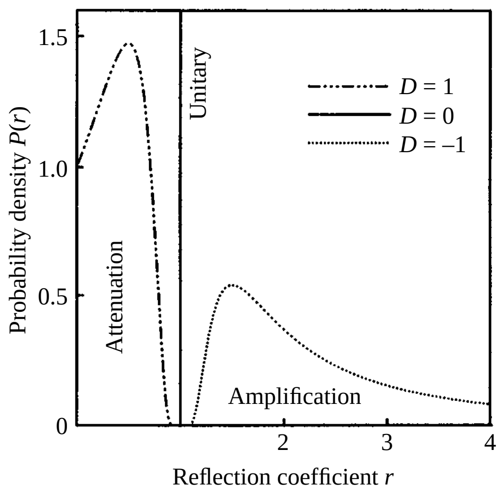

Fig. 1. Limiting probability distribution $P(r)$ in the limit of infinite length for coherent absorption ($D = 1$), unitary ($D = 0$), and for coherent amplification ($D = -1$).

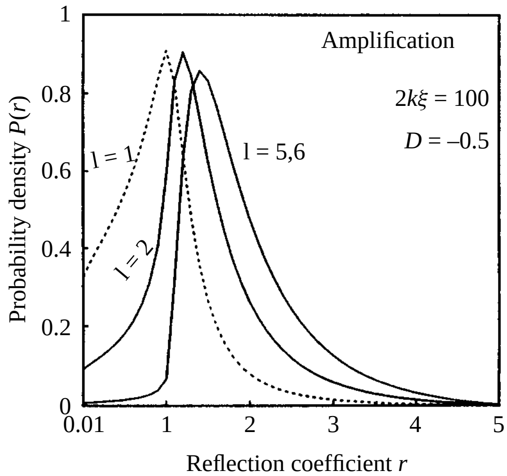

Fig. 2. Evolution of probability distribution $P(r)$ with the sample length ℓ, for fixed weak disorder $2k\xi = 100$, and coherent amplification $D = -0.5$.

as $(-i\epsilon_0'')$ does not represent a matched reflectionless amplifier, the resulting gain is not a monotonically increasing function of ϵ_0''. This point has received some attention [27], and can lead to some subtle effects [28]. Also, so far we have introduced randomness in the real part of the dielectric constant only. It appears, however, that a random amplification $(-i\epsilon_0'')$-only system is qualitatively different and is currently under study. There is also the fundamental question of mode selection here when the virtual cavity is provided by localization. We think this is essentially the result of the interplay of localization and coherent amplification in the presence of nonlinearity, i.e., the light-intensity dependence of ϵ_0'' that has been neglected in the present linear analysis.

Finally, we would like to report briefly on a rather curious result obtained in a recent experiment [20] where we have investigated experimentally the random laser action in a dilute aqueous suspension of polystyrene microspheres containing the pumped laser dye rhodamine 590. The low refractive index contrast of the water/polystyrene system and the low concentration of scatterers gave an estimated meanfree path much greater than the sample size. However, even for this sub-meanfree path sample we obtained a laser action evidenced by the gain narrowing observed, which, of course, was absent for the pure dye solution, i.e., without the scatterers. Clearly, we cannot here invoke diffusion, much less the localization. Instead, we consider this as possible evidence for the otherwise ignorable sub-meanfree path Poissonian tail events which now become effective due to coherent amplification. There may be life before the meanfree path!

To conclude, the interplay of localization and coherent amplification in a random active medium poses an interesting problem involving interplay of two coherence phenomena–localization due to coherent back-scattering and coherent amplification: it may even be important. One speaks of turbid lasers and photonic paints [16, 19]. It is currently an active area of research, experimental as well as theoretical.

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