Recap

5. b) Many-to-one markets: substitutable preferences

6. a) Large market results: empirical evidence
Outline

6. b) **Large market results**: incentives, couples
Part 6: Large Market Results.
Theoretical Result

**Theorem.** Even allowing women *arbitrary* preferences, the fraction of agents with more than one stable mate tends to zero as $n$ tends to infinity (holding $k$ fixed).

[Immorlica-Mahdian ‘05, Kojima-Pathak ‘09]
Intuition

Doctors

Hospitals

$q_1 > 3$
$q_2$
$q_3 = 1$
One-to-One Markets

Proof Sketch:

1. An algorithm that counts the number of stable husbands of a given woman.
2. Bounding probability of having $> 1$ stable husband in terms of the number of singles.
3. Bounding the number of singles by the solution of the occupancy problem.
Step 1: Finding Stable Husbands of $g$

- Use men-proposing to find stable matching
- Whenever algorithm finds stable matching,
  - have $g$ divorce husband and truncate preference
  - continue men-proposing algorithm
- Terminate whenever
  - previously married man runs through his list, or
  - previously single woman receives a proposal
**Question.** If each woman has an arbitrary complete preference list, and each man has a random list of $k$ women, what is the probability that this algorithm returns more than one stable husband for $g$?

The main tool that we will use to answer this question is the *principle of deferred decisions*:

Men do not pick the list of their favorite women in advance; Instead, every time a man needs to propose, he picks a woman at random and proposes to her. A man remains single if he gets rejected by $k$ different women.
Step 2: Bounding the Probability

• Consider first stable matching $\mu$ found by alg.
• Let $A_\mu = \{\text{single women in } \mu\}$, and $X_\mu = |A_\mu|$.
• Conditioning on random choices made before algorithm finds $\mu$,

\[
    \Pr[g \text{ has } > 1 \text{ stable mate } | \mu] < \frac{1}{(X_\mu+1)}
\]

\[
    \Pr[g \text{ gets another proposal } | \mu] = \frac{1}{(X_\mu+1)}
\]

• Removing conditioning, prob. < $E_\mu \left[ \frac{1}{(X_\mu+1)} \right]$
Step 3: Number of Singles

- Want to compute $E_\mu \left[ \frac{1}{(X_\mu + 1)} \right]$
- Note probability woman remains single is at least probability she’s never named by a man.
- Let $Y_{m,n} = \#$ empty bins when $m$ balls thrown randomly into $n$ bins.

Lemma. $E_\mu \left( \frac{1}{(X_\mu + 1)} \right) \leq E_\mu \left( \frac{1}{(Y_{(k+1)n,n} + 1)} \right) + \frac{k^2}{n}$
The Occupancy Problem

Lemma. \[ E_\mu \left( \frac{1}{(Y_{m,n}+1)} \right) \leq \frac{e^{m/n}}{n} \]

Proof Sketch.

• Use the principle of inclusion and exclusion to compute \( E[1/(Y_{m,n}+1)] \) as a summation.

• Compare this summation to another (known) summation term-by-term.
Putting it all together...

**Theorem.** In the model where women have arbitrary complete preference lists and men have random lists of size $k$, the probability that a fixed woman has more than one stable husband is at most

$$\frac{e^{k+1} + k^2}{n}.$$
Extensions

Arbitrary IID distributions?
Extensions

Women: n hospital positions, preference is a uniform random permutation of all men

Men: n applicants, preference chosen by:
  • distribution $D$ over women
  • construct list iteratively by selecting from $D$
Extension

Theorem. In the above model, the probability that a fixed woman has more than one stable husband is at most:

\[
\frac{16k}{\ln(n)} + \frac{3\ln(n)}{4k \sqrt{n}} = O\left(\frac{k}{\ln(n)}\right)
\]
Many-to-One Extension

**Theorem.** Truthfulness is almost surely a best response when others are truthful.

**Proof Sketch:**

- Argue “dropping strategies” comprehensive
- Modify alg counting stable husbands to study rejection chains of dropping strategies
- Argue rejection chains don’t return to college w/high prob. when market large