Market Design: Lecture 4

NICOLE IMMORLICA, NORTHWESTERN UNIVERSITY
OH, BY THE WAY, TAJEL, BEFORE YOU GO...

ANOTHER UNIVERSITY HAS OFFERED ME A POSITION AS CHAIR OF THEIR DEPARTMENT.

WHAT?

I HAVEN'T DECIDED IF I'M GOING OR NOT, BUT I THOUGHT YOU SHOULD KNOW.

WAIT, WAIT, LET ME GET THIS STRAIGHT...

YOU'RE LEAVING??

OR NOT. IT'S NOT A BIG DEAL. MAYBE I SHOULDN'T HAVE TOLD YOU.
Recap

4. b) Incentives: complete information Nash equilibria, incomplete information

5. a) Many-to-one markets: responsive preferences
Outline

5. b) Many-to-one markets: substitutable preferences
6. Large market results: incentives, couples
Part 5: Many-to-one Markets.
Matching with Contracts

- doctors $D = \{d_1, \ldots, d_n\}$
- hospitals $H = \{h_1, \ldots, h_p\}$
- contracts $X = \{x\}$, each containing one doctor $x_D$ and one hospital $x_H$
  - college admissions: $X = D \times H$
  - worker-firm (Kelso-Crawford): $X = D \times H \times W$ for discrete set of wages $W$
Matching with Contracts

• doctor $d$ can sign at most one contract, pref. $P(d)$ given by total order on \( \{x : x_D = d\} \)

• hospital $h$ has preferences over sets of contracts, each doctor appears at most once
Choice Sets

• Func. $C_a(X')$ outputs subset of $X'$ that a prefers

• For any subset $X'$ of $X$,
  - $C_d(X')$ in $\{x' \in X' : x'_D = d\} \cup \{null\}$
  - $C_d(X') >_d x$ for all other $x$ in $X' \cup \{null\}$

• For any subset $X'$ of $X$,
  - $C_h(X') \subseteq \{x' \in X' : x'_h = h\} \cup \{null\}$
  - $\{x, y\} \subseteq C_h(X')$ implies $x_D \neq y_D$
  - $C_h(X') >_h S$ for any other subset $S$ in $X' \cup \{null\}$
Notation

- chosen set for doctors $C_D(X') = \cup_d C_d(X')$
- chosen set for hospitals $C_H(X') = \cup_h C_h(X')$
- rejection set for doctors $R_D(X') = X' - C_D(X')$
- rejection set for hospitals $R_H(X') = X' - C_H(X')$
Stable Allocations

Set of contracts $X'$ stable if

- feasible: each doctor appears at most once
- individually rational: $C_H(X') = C_D(X') = X'$
- no blocking coalitions: $(h, X^*)$ block $X'$ if
  - $C_h(X') \neq X^*$
  - $C_h(X' \cup X^*) = X^*$
  - $C_D(X' \cup X^*)$ contains $X^*$
Characterization of Stability

• $X'$ stable iff any alternative contract would be rejected by some doctor or hospital

• opportunity sets: currently considering $Z$
  – available to hospitals: $X - R_H(Z)$
  – available to doctors: $X - R_D(Z)$
Characterization of Stability

1. hospitals faced with options
2. hospitals reject some of their options
3. other options plus unproposed contracts are opportunities
4. when doctors face these opportunities, they must rule out unproposed contracts
Fixed Point Theorem

Theorem. If \((X_D, X_H)\) is solution to
\[
X_D = X - R_H(X_H) \\
X_H = X - R_D(X_D)
\]
then the intersection of \(X_H\) and \(X_D\) is stable.

Conversely, if \(X'\) stable, there exist \(X_H\) and \(X_D\)
whose intersection is \(X'\).
Substitutability

Defn. Choice function $C_a(.)$ **substitutable** if for all contracts $x$, $z$ and subsets $X'$,

$z$ not in $C_a(X' \cup \{z\})$ implies $z$ not in $C_a(X' \cup \{x,z\})$

or, equivalently, $R_a(.)$ monotone.

Intuition: Receiving new offers makes agent weakly less-interested in old offers.
Substitutability

**Example:** contracts with wages: \( X = D \times H \times W \)

- demand theory substitutes: if at wage vector \( v_w \), \( h \) chooses \( d \), then \( h \) still chooses \( d \) at \( v'_w \),
  - \( v'_w(d) = v_w(d) \)
  - \( v'_w(d') \geq v_w(d') \) for all \( d' \neq d \)

- choice function substitutable iff satisfies demand theory substitutes
Tarski’s Fixed Point Theorem

Defn. Given lattice \((A, \geq)\), function \(f: A \to A\) is \textit{isotone} if monotone on partial order:

\[ a \geq b \text{ implies } f(a) \geq f(b) \]

Theorem. Any isotone function \(f\) on a complete lattice has a fixed point, and the set of fixed points form a complete lattice with respect to \(\geq\).
Existence

• Define lattice \((X \times X, \geq)\) where
  \[ (X_D, X_H) \geq (Y_D, Y_H) \text{ iff } Y_D \subseteq X_D \text{ and } X_H \subseteq Y_H \]

• Complete since \((X, \emptyset)\) and \((\emptyset, X)\) max/min elts

• Define \(F(X_D, X_H) = (F_1(X_H), F_2(F_1(X_H)))\) where
  – \(F_1(X') = X - R_H(X')\) and \(F_2(X') = X - R_D(X')\)
  – so \(X' = X_H\) are contracts currently on the table
  – \(F_1(X_H)\) everything hospitals don’t reject
  – \(F_2(F_1(X_H))\) what doctors pass back to hospitals
Existence

Claim. If prefs substitutable, then F isotone.

Prf. Follows from monotonicity of $R_h(.)$.

Interpretation. DA is repeated iterations of F,

- hospital-proposing, start at min elt ($X_D = \emptyset, X_H = X$)
- doctor-proposing, start at max elt ($X_D = X, X_H = \emptyset$)
Example: Hospital-Proposing

\[
\begin{align*}
P(d_1) &= h_1, h_2 & P(h_1) &= d_1, d_2 \\
P(d_2) &= h_1, h_2 & P(h_2) &= \{d_1, d_2\}, d_1, d_2
\end{align*}
\]

Initialize \( X_D = \emptyset, X_H = X = \{(h_1, d_1), (h_1, d_2), (h_2, d_1), (h_2, d_2)\} \)

- Hospitals reject: \( R(X_H) = \{(h_1, d_2)\} \)
- Hospitals offer: \( X_D = \{(h_1, d_1), (h_2, d_1), (h_2, d_2)\} \)
- Doctors reject: \( R(X_D) = \{(h_2, d_1)\} \)
- Doctors choose: \( X_H = \{(h_1, d_1), (h_1, d_2), (h_2, d_2)\} \)
- Hospitals reject: \( R(X_H) = \{(h_1, d_2)\} \)

Output \( X_D \cap X_H = \{(h_1, d_1), (h_2, d_2)\} \).
Application: Unsplittable Flow

- jobs $J = \{j_1, ..., j_n\}$
  - job $j$ has size $s(j)$
  - preference list $P(j)$ over machines
- machines $M = \{m_1, ..., m_p\}$
  - machine $m$ has capacity $c(m)$
  - preference list $P(m)$ over individual jobs
  - preferences over sets responsive
- example: match groups of friends to sports teams without splitting up a group.
Application: Unsplittable Flow

• allocation $\mu$ assigns jobs to machines
  – strictly feasible: $\sum_{j \in \mu(m)} s(j) \leq c(m)$
  – stable: IR and no pair $(j, m)$ s.t. $m > j \mu(m)$ and for some $j'$ in $\mu(m)$, $s(j) \leq c(m) - \sum_{k \neq j'} \in \mu(m) s(k)$, $j > m j'$
• define $C_m(X)$ to be top contracts that “fit”
• then $C_m(X)$ substitutable, so stable allocations exist (must check defn’s of stability match)
Application: Unsplittable Flow

- allocation $\mu$ assigns jobs to machines
  - weakly feasible: $\Sigma_{j \in \mu(m)} s(j) \leq c(m) + \max_j(s(j))$
  - stable: IR and no pair $(j, m)$ s.t. $m >_j \mu(m)$ and for some $j'$ in $\mu(m)$, $\Sigma_{k \neq j' \in \mu(m)} s(k) < c(m)$, $j >_m j'$
- now $C_m(X)$ is top contracts that just “overfit”
- by same argument, still substitutable
- stable allocations exist
Part 6: Large Market Results.
Entry-Level Labor Markets

Case Study:

National Residency Matching Program (NRMP): physicians look for residency programs at hospitals in the United States
## A Brief History of NRMP

### Case Study:

<table>
<thead>
<tr>
<th>1950</th>
<th>1990</th>
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<tbody>
<tr>
<td>decentralized,</td>
<td>centralized clearinghouse,</td>
</tr>
<tr>
<td>unraveling,</td>
<td>95% voluntary participation</td>
</tr>
<tr>
<td>inefficiencies</td>
<td>dropping participation</td>
</tr>
<tr>
<td></td>
<td>sparks redesign to accommodate couples,</td>
</tr>
<tr>
<td></td>
<td>system still in use</td>
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</table>
NRMP Theory and Practice

Theory:

Gale-Shapley stable marriage algorithm: NRMP central clearinghouse algorithm corresponds to deferred acceptance algorithm (at first hospital-, and then student-proposing)
NRMP Redesign

What were the issues?

1. NRMP favored hospitals.
   Hospital-proposing deferred acceptance produces hospital-optimal matching.

2. NRMP was manipulable.
   Both students and hospitals have incentives to report false preferences.
Match Variations

Couples.
Married students have joint preferences over geographically close positions.

Reversion.
Hospital programs may wish to revert unfilled positions to other programs at same hospital.
Problems with Match Variations

What were the issues?

3. Algorithm choice affects unmatched agents. Not with no match variations (rural hospital theorem), but possible otherwise.

4. There may be no stable matching. Stable matching exists with no match variations, but may not otherwise.
Concerns with NRMP

1. NRMP favored hospitals.
2. NRMP was manipulable.
3. Algorithm choice affects unmatched agents.
4. There may be no stable matching.

... empirical study.

[Roth-Peranson ‘99]
# Descriptive Statistics of NRMP

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<tbody>
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<td><strong>Applicants</strong></td>
<td></td>
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<td></td>
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<tr>
<td># with ROL</td>
<td>20071</td>
<td>20916</td>
<td>22353</td>
<td>22937</td>
<td>24749</td>
</tr>
<tr>
<td># couples</td>
<td>694</td>
<td>854</td>
<td>892</td>
<td>998</td>
<td>1008</td>
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<td><strong>Programs</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td># with ROL</td>
<td>3170</td>
<td>3622</td>
<td>3662</td>
<td>3745</td>
<td>3758</td>
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<tr>
<td># positions</td>
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<td>22737</td>
<td>22801</td>
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## Difference between DA Algorithms

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<td>20</td>
<td>16</td>
<td>20</td>
<td>14</td>
<td>21</td>
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<td>9</td>
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<tr>
<td>prefer applicant-proposing</td>
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<td>16</td>
<td>11</td>
<td>14</td>
<td>12</td>
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<td>new matched</td>
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<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>new unmatched</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
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<td>15</td>
<td>23</td>
<td>15</td>
<td>19</td>
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<tr>
<td>prefer hospital</td>
<td></td>
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<td>15</td>
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<td>14</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>new unmatched</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>
Bounding Potential Manipulations

**Theorem.** Equilibria produce stable matchings.

**Corollary.** An agent can manipulate only if he or she has more than one stable mate.
Bounding Potential Manipulations

**Theorem.** An agent’s best stable mate is the one he or she receives when proposing (and worst when not proposing).

**Corollary.** An agent has more than one stable mate if and only if he or she receives different mates at the men-proposing and women-proposing algorithms.
## Bounding Potential Manipulations

<table>
<thead>
<tr>
<th>Year</th>
<th># applicants</th>
<th># applicants who could manipulate</th>
<th># positions</th>
<th># programs</th>
<th># programs who could manipulate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1987</td>
<td>20071</td>
<td>20</td>
<td>19973</td>
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<td>1994</td>
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<td>1996</td>
<td>24749</td>
<td>21</td>
<td>22578</td>
<td>3758</td>
<td>19</td>
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</tbody>
</table>
Explanations

What limits number of stable mates?

1. Preferences are correlated.
   Applicants agree on prestigious hospitals; hospitals agree on promising applicants.

2. Preferences are short.
   Applicants typically list at most 15 hospitals.
A Probabilistic Model

**Women**: $n$ hospital positions, preference is a uniform random permutation of all men

**Men**: $n$ applicants, preference chosen uniformly at random from lists of at most $k$ women
A Probabilistic Model

Conjecture [Roth-Peranson ‘99]. Holding k constant as n tends to infinity, the fraction of women with more than one stable mate tends to zero.

The potential to manipulate is vanishingly small.
Simulating that there are substantial opportunities for strategic manipulation but that these have been exhausted by the time we look at the ROIs submitted to the match, because the participants have already behaved strategically in an optimal way. Another counterhypothesis could be that the hybrid nature of the preexisting NRMP algorithm in fact produces matches that are far from the worst possible stable matching for applicants, and that the set of stable matchings is therefore substantially larger than we detect. The results discussed in this section show that these hypotheses are implausible, because when we looked at similarly sized artificial matches, in which we can examine the hypothetical participants' true preferences, we find that the set of stable matchings is close to the size we have computed from the ROI data. Thus the study of simple markets provides an explanation of not only the direction of the effects we have been examining, but also their small size.

VII. Theory and Computation in Economic Design: Some Methodological Reflections

Perhaps the first rule of any design effort is that "details matter." The details determine what outcomes are even feasible, and so they matter in the most basic aspects of design; and they have implications for all of the market's properties, so they matter for the subtlest aspects of the design's consequences. Thus, every design effort will be different. But if we are to develop a body of knowledge about design practice in economics, we need to think about the methodological issues that may be common to many design efforts. This section is an attempt to put the methodological issues encountered in the NRMP design and evaluation into a context that may be useful for other design efforts. Specifically, this design effort involved the continual interplay among various aspects of simple theory, computational experiments, and theoretical computation. The simple theory guided the design of computational experiments on the complex system, which provided unpredictable results that were then explained by theoretical computation.

The reason why there are gaps between theory and design is that, just as design is detailed, theoretical models must often be sparse, to be useful for organizing and directing work in a variety of applications whose connections may become apparent only with the benefit of the references.

Note: $C(n)$ is the number of applicants who get different stable matches, when the market size is $n$.  

**Figure 1.** Size of the set of stable matchings as a fraction of $n$, when $k = n$.  

Fraction of agents with more than one stable mate when $k = n$. 

**References**
SimulaEon

FracEon with more than one stable mate when k constant, n grows.

Much of this paper has therefore been concerned with filling the gaps between simple abstract markets and complex real ones. But before we discuss the filling of gaps, it is useful to recall the essential role played by the theory of simple matching markets. This role ranged from suggesting the basic design of the clearinghouse algorithm and the comparisons of the algorithms, to directing attention to aspects of the market in which problems might be anticipated, and to offering insights into how these might be overcome.

It was the existing simple theory, and the empirical studies it permitted to be conducted on field data, that pointed to the importance of stable matchings. Although counterexamples showed that stable matchings might not exist in the complex American medical market (Roth, 1984), the theory of simple markets suggested a general architecture for an algorithm to find stable matchings. Furthermore, it showed that algorithms in which proposals were issued by applicants could be expected to produce stable matchings as favorable as possible to applicants. In short, the body of theory that existed prior to the start of this design (e.g., as summarized in Roth and Sotomayor [1990]) already constituted a rough road map for the mechanism design and evaluation reported here.

At the same time, the existing body of theory, through counterexamples designed to explore its limits (inspired by empirical studies of existing markets), pointed to questions that needed to be answered. These included the role of sequencing in design of the algorithm, the frequency with which the algorithm might fail to find a stable matching, and the frequency with which opportunities for strategic manipulation might arise. These all required estimations of magnitudes, which in turn required computational experiments on the data. Some of these computational experiments were straightforward to conduct. But for estimating how often strategic opportunities might arise, the theory

\[ C(n) = \frac{\text{number of applicants who get different stable matches}}{n}, \text{for different values of } k \text{ (uncorrelated programs on an applicant's ROL).} \]

Figure 2: Size of the set of stable matchings as a fraction of n for different values of k (uncorrelated programs on an applicant's ROL).

Fraction with more than one stable mate when k constant, n grows.
Theoretical Result

**Theorem.** Even allowing women *arbitrary* preferences, the fraction of agents with more than one stable mate tends to zero as $n$ tends to infinity (holding $k$ fixed).

[Immorlica-Mahdian ‘05, Kojima-Pathak ‘09]
Economic Implications

1. When others are truthful, almost surely an agent’s best strategy is to tell the truth.

2. There is an equilibrium of women-proposing DA in which \( (1-o(1)) \times n \) agents are truthful.

3. In settings of incomplete information, there is a \( (1+o(1)) \) approximate Bayes-Nash equilibrium in which all agents are truthful.