Recap

**Theorem 0.1 (Tutte-Berge Formula):** For any graph $G$, $\nu(G) = \min_{U \subseteq V} (|V| + |U| - o(G - U))/2$.

**Def:** $U$ is a Tutte-Berge witness if $\nu(G) = (|V| + |U| - o(G - U))/2$.

**Def:** The Edmonds-Gallai decomposition partitions the vertices $V$ of a graph $G$ into sets

- $D(G)$ – set of vertices $v$ such that $v$ is exposed by some maximum matching,
- $A(G)$ – set of neighbors of $D(G)$, and
- $C(G)$ – set of all remaining vertices.

Construction: vertices reachable by odd/even alternating paths from a vertex $v \in X$.

Let $M$ be matching returned by Edmonds’ Algorithm, $X$ be exposed vertices.

- Even := $\{v : \exists$ even alternating path from $X$ to $v\} = D(G)$, odd compoents in $G - U$ and factor critical
- Odd := $\{v : \exists$ odd alternating path from $X$ to $v$ and no even one$\} = A(G)$

**Claim:** There is no edge between Even and Free.

**Claim:** There is no edge within Even in $G_0$.

**Claim:** $C(G)$ is even components.

**Proof:** We proved no edge between Even and Free, so $M$ matches vertices of $C(G)$ to vertices of $C(G)$ so $|M \cap E(C(G))| = |C(G)|/2$.

**Claim:** $D(G)$ is odd components, each of which is factor-critical.

**Proof:** For every connected component $H$ of $(G - U) \cap D(G)$, we show:

1. Either $|X \cap H| = 1$ and $|M \cap \delta(H)| = 0$, or $|X \cap H| = 0$ and $|M \cap \delta(H)| = 1$ (where $\delta(H)$ is edges with exactly one endpoint in $H$).
2. $H$ is factor-critical.

**Tutte-Berge Witnesses**

**Theorem 0.2** $U = A(G)$ is a Tutte-Berge witness.

**Proof:** Want to show

$$|M| \geq \frac{1}{2}(|V| + |A(G)| - o(G \setminus A(G)))$$

(other direction always holds). Note that

$$|M| \geq |M \cap E(C(G))| + |M \cap E(D(G))| + |M \cap \delta(A(G))|$$

and

- we showed $|M \cap E(C(G))| = |C(G)|/2$
- previous proof, first subclaim, showed $|M \cap E(D(G))| = \frac{1}{2}(|D(G)| - o(G \setminus A(G)))$ (each component leaves one unmatched or matched to outside)
- $|M \cap \delta(A(G))| = |A(G)|$ since all $v \in A(G)$ matched to vertices of $D(G)$ (if not can grow matching)

so have

$$\frac{1}{2} \left( |C(G)| + |D(G)| + 2|A(G)| - o(G \setminus A(G)) \right)$$

$$= \frac{1}{2} \left( |V| + |A(G)| - o(G \setminus A(G)) \right)$$

as claimed.

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**Matching Polytope**

**Def:** For a matching $M \subseteq E$, define its incidence vector $\chi(M) \in \mathbb{R}^{|E|}$ to be $\chi(M)_e = 1$ if $e \in M$, 0 otherwise. The **matching polytope** $\mathcal{P}$ is the convex hull of incidence vectors of matchings.

**Goal:** Represent $\mathcal{P}$ by set of linear inequalities on variables $\{x_e\}$.

**Question:** Come up with some inequalities.

- $x_e \geq 0$
- $x(\delta(v)) = \sum_{e \in \delta(v)} x_e \leq 1$: each vertex has at most one adjacent edge

Call this polytope $P_1$.

**Note:** $\mathcal{P} \subseteq P_1$

**Example:** $P_1$ is not contained in $\mathcal{P}$: triangle

- $\mathcal{P} = \text{conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$
- $(0.5, 0.5, 0.5) \in P_1$ but not in $\mathcal{P}$

**Question:** Additional constraint?

**Def:** The **blossom constraints** are

$$x(E(U)) = \sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}, U \subseteq V, |U| \text{ odd.}$$

The polytop $P_2$ is $P_1$ together with the blossom constraints.

**Theorem 0.3** (Edmonds, 1965): $P_2$ equals the matching polytope $\mathcal{P}$.

[[Edmonds gave algorithmic proof, we use TDI.]]
Total Dual Integrality

Recall primal/dual LPs:

Primal $P$:
max $c^T x$ s.t. $Ax \leq b$

Dual $D$:
min $b^T y$ s.t. $A^T y = c$ and $y \geq 0$

Def: A linear system $\{Ax \leq b\}$ is totally dual integral (TDI) if for any integral cost vector for the primal such that $c^T x, Ax \leq b$ is finite, there exists an integral optimal dual solution.

**Theorem 0.4** (Edmonds-Giles, 1979): If a system $\{Ax \leq b\}$ is TDI and $b$ is integral, then $\{Ax \leq b\}$ is integral (i.e., the extreme points are integral).

[[We will prove this later.]]

**Note:** We will show $P_2$ is TDI and hence is convex hull of all integral points contained in it, proving that $P_2 = \mathcal{P}$.

Polyhedral combinatorics:

- define $Ax \leq b$ and show integral with vertices corresponding to certain combinatorial objects.
- show system is TDI so dual has integral solution as well.
- find combinatorial interpretation for dual to get min-max theorem, or also helps design primal-dual alg by discretizing space.

[[Rational polyhedra have TDI representations.]]

**Theorem 0.5** (Giles-Pullyblank, 1979): For a rational polyhedron $\mathcal{P}$, there exist $A$ and $b$ with $A$ integral such that $\mathcal{P} = \{x : Ax \leq b\}$ and the system is TDI.

Note: $b$ integral iff $\mathcal{P}$ integral

**Example:**

$\mathcal{P} = \text{conv}\{(0,3), (2,2), (0,0), (3,0)\}$

Representation: $\{x, y : x \geq 0, y \geq 0, x + 2y \leq 6, 2x + y \leq 6\}$

Draw figure.

Suppose $c = (1,1)$. Primal opt is $(2,2)$ and tight constraints are $(1,2)$ and $(2,1)$.

[[Tight constraints are of $A$, i.e., normals of facets at $(2,2)$]]

Thus for $A^T y = c$ to have integer solution, must be able to write $c$ as integer combination of $(1,2)$ and $(2,1)$.

[[Tight constraints in opt primal soln are non-zero variables in opt dual soln.]]

**Question:** Make TDI with new representation?

Representation: add inequalities $x + y \leq 4, x, y \leq 3$, becomes TDI.

Hilbert Basis

**Question:** When is a system TDI? Consider problem $\max\{cx : Ax \leq b\}$ with $c$ integral and opt soln $\beta < \infty$.

- There’s opt soln $x^*$ in some face $F$ defined by $\{Ax \leq b\}$ and $cx = \beta$.
- Suppose $F$ is an extreme point, let $A'x = b'$ be inequalities tight at $x^*$ (i.e., $A'x^* = b'$).
- Dual is $\min\{b^T y : A^T y = c, y \geq 0\}$ so opt dual corresponds to $c$ being expressible as non-neg combination of row vectors, i.e., the cone of row vectors of $A'$.
- For $y$ to be integral, must be able to ex-
press points in cone as integer combinations.

**Def:** A set of vectors \( \{a_i : a_i \in \mathbb{Z}^n \} \) is a Hilbert basis if for any integral \( c \in \text{cone}(a_i) = \{ \sum_i \lambda_i a_i : \lambda_i \geq 0 \} \), there exist non-negative integers \( \mu_i \) such that \( c = \sum_i \mu_i a_i \).

**Example:** For vertex \((3, 0)\) above, tight constraints \( \{(1, 2), (-1, 0), (0, 1)\} \) form a Hilbert basis. 

\[
\lambda_1 - \lambda_2 = c_1 \quad \text{and} \quad 2\lambda_1 + \lambda_3 = c_2
\]

so for \( \lambda_1 > 0, \ 2\lambda_1 + \lambda_3 \geq 2 \) and we can get all these. For \( \lambda_1 = 0, \lambda_2, \lambda_3 \) are non-neg integers if \( c \) integral, so we can get all these too.

**Theorem 0.6** The rational system \( Ax \leq b \) is TDI iff for each face (actually sufficient to check for each extreme point), tight constraints form a Hilbert basis.

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\text{[F. by above observations, i.e., LP- duality.]} \]

We can always add constraints to make it TDI:

**Theorem 0.7** Any rational polyhedral cone \( C = \{ \sum_i \lambda_i a_i : \lambda_i \geq 0, \lambda_i \in \mathbb{R} \} \) with \( \{a_i\} \) integral has a finite integral Hilbert basis.

**Proof:** Let \( Q = \{ \sum_i \lambda_i a_i : 0 \leq \lambda_i \leq 1 \} \) and note for any integral \( c \in C \),

\[
c = \sum_i \lambda_i a_i = \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i + \sum_i \lfloor \lambda_i \rfloor a_i
\]

Call this \( z + w \). Note

- \( w \) integral since \( a_i \) and \( \lfloor \lambda_i \rfloor \) are
- \( c \) integral by assumption hence \( z \) is too
- \( z \in Q \)
- \( a_i \in Q \)
- thus \( w \) integral combination of integral vectors in \( Q \)
- so \( c = z + w \) is also integral combination of integral vectors in \( Q \)

and therefore \( Q \cap \mathbb{Z}^n \) is a finite integral Hilbert basis for \( C \).

**Note:** In fact don’t need to assume \( \{a_i\} \) integral, follows from rationality of cone.

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\text{[[We are now ready to prove main theorem.]]}
\]

**Claim:** (Edmonds-Giles, 1979): If a system \( \{Ax \leq b\} \) is TDI and \( b \) is integral, then \( \{Ax \leq b\} \) is integral.

**Proof:** By contradiction.

- Consider extreme point \( x^* \) of \( P \) s.t. \( x^*_j \not\in \mathbb{Z} \) for some \( j \).
- Let \( c \) be integral vector s.t. \( x^* \) unique opt by picking rational vector in cone at \( x^* \) and scaling.
- Consider \( \hat{c} = c + \frac{1}{q} e_j \) (inside cone for large enough \( q \)).
- Since \( q\hat{c}^T x^* - qc^T x^* = x^*_j \not\in \mathbb{Z} \), either \( q\hat{c}^T x^* \) or \( qc^T x^* \) not integral.
- By duality and fact that \( b \) is integral, one of corresponding dual soln \( \hat{y} \) or \( y \) not integral.
- Contradicts TDI since both \( q\hat{c} \) and \( qc \) integral.