Sets Review

Universe $U = \{0, 1, 2, 3\}$ with bit vector $x = x_0 x_1 x_2 x_3$.

Sets $A = \{1, 2\}, x_A = 0110$, and $B = \{2, 3\}, x_B = 0011$.

Set concepts:

- union $A \cup B = \{1, 2, 3\}, x_{A \cup B} = 0111$
- intersection $A \cap B = \{2\}, x_{A \cap B} = 0010$
- complement $A^c = \{0, 3\}, x_{A^c} = 1001$

Induction Review

Basic Induction:

Want to prove $P(n)$.

- Prove base case $P(1)$.
- Prove $P(n) \rightarrow P(n+1)$ (by direct proof).
  - Inductive hypothesis: assume $P(n)$.
  - Inductive step: using hypothesis, derive $P(n+1)$.

Invariants by Induction

[[Useful to prove algorithm is correct.]]

Example: Robot moves on diagonals of grid, starting at $(0, 0)$.

Claim: Robot never steps on flower at $(0, 1)$.

States after

- 1 move: $(1, 1), (1, -1), (-1, 1), (1, 1)$
- 2 moves: $(0, 0), (0, 2), (2, 2), (2, 0), \ldots$
- etc.

Sum of coordinates always even!

Predicate $P(t)$: After $t$ steps, if robot is at $(x,y)$, then $x+y$ is even.

Claim: Sum of coordinates always even.

Proof: By induction.

- Base case: $P(0)$ is true since starting position $(0, 0)$ is $0 + 0 = 0$ is even.
- Inductive hypothesis: after $t$ steps, robot is at $(x,y)$ where $x + y$ is even.
- Inductive step: by cases.
  - Robot moved northwest. New position is $(x-1, y+1)$. Sum is $x + y$, even by hypothesis.
  - Robot moved northeast. New position is $(x+1, y+1)$. Sum is $x+y+2$, even.
Since $1 + 0 = 1$ is odd, robot never steps on flower.

**Example:** The 8-puzzle: slide tiles to convert

```
A B C
D E F
H G .
```

into

```
A B C
D E F
G H .
```

**Claim:** Not possible.

**Note:** Row moves don’t change order.

**Note:** Column moves change order of two pairs.

**Def:** Tiles $T_1$ and $T_2$ are inverted if out-of-alphabetical order.

```
A B C
D E F
E H .
```

Has three inversions: $(D, F)$, $(E, F)$, $(E, G)$.

**Claim:** Moves change number of inversions by 2 or 0.

**Proof:**

- Row move doesn’t change number.

- Column moves switch exactly two pairs:
  - If both pairs originally inverted, total number of inversions decreases by 2.
  - If just one pair originally inverted, it gets sorted and other gets inverted, total doesn’t change.

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**Claim:** In every configuration reachable by legal moves, parity of number of inversions is odd (i.e., sum is an odd number).

**Proof:** By induction.

- Base case: initial configuration has 1 inversion.

- Inductive hypothesis: after $t$ moves, odd parity.

- Inductive step: by above claim, number changes by 2 or 0, so $t + 1$’th move has odd parity by inductive hypothesis.

Sorted board not reachable since parity is even.

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**Strong Induction**

Useful when predicate $P(n + 1)$ naturally depends on some $m < n$.

Suppose you want to prove $P(n)$.

- Prove base case $P(1)$.

- Inductive hypothesis: assume $P(m)$ for all $1 \leq m \leq n$.

- Inductive step: using hypothesis, derive $P(n + 1)$.

**Example:** Prime factorization.

**Claim:** Every integer $n > 1$ is product of primes.

**Proof:** By strong induction.

- Base case $P(2)$: $2 = 1 \times 2$ is product of primes.
• Inductive hypothesis: \( m \) is product of primes for all \( 2 \leq m \leq n \).

• Inductive step:
  - If \( n + 1 \) prime, done.
  - If not, then \( n + 1 = km \) for some integers \( k, m \in \{2, 3, \ldots, n\} \).
  - By inductive hypothesis, \( k, m \) are products of primes, and thus so is \( n + 1 \).

**Example:** Making change.

**Claim:** Every amount of postage of 12 cents or more can be formed using just 4 and 5 cent stamps.

**Proof:** By strong induction

- \( P(n) = n \) cents of postage formed with 4, 5 cent stamps
- \( P(n) \) true for \( n \in \{12, 13, 14, 15\} \)
- assume \( P(k) \) for all \( k \leq n \)
- \( P(n + 1) \): use IH to get \( n - 3 \) cents of postage and add a 4 cent stamp

\[ \square \]

**Claim:** It takes at most \( nm - 1 \) breaks to divide an \( n \)-by-\( m \) chocolate bar.

**Proof:**

- By strong induction on number \( k \) of squares in bar.
- Base case: With 1 square, need \( 1 \cdot 1 - 1 = 0 \) breaks.
- Inductive hypothesis: Assume any bar with at most \( k \) squares can be divided with \( k - 1 \) breaks.

**Inductive step:**

- Given a bar with \( k + 1 \) squares, use one break to get two bars with \( s_1 \) and \( s_2 \) squares respectively where \( s_1 + s_2 = k + 1 \).
- Use inductive hypothesis to break these with \( s_1 - 1 \) and \( s_2 - 1 \) breaks respectively.

So used \( 1 + (s_1 - 1) + (s_2 - 1) = s_1 + s_2 - 1 = (k + 1) - 1 \) breaks.

\[ \square \]

**Structural Induction**

Induction on recursively-defined data types.

**Example:** parentheses.

**Def:** Set \( M \) of matched parenthetical statements:

- empty string \( \lambda \) is in \( M \)
- if \( s, t \in M \), then \( (s)t \in M \)
- etc.

**Template:**

- Prove for base cases of definition.
- Prove for constructor case assuming holds for component types.
Claim: \( \forall s \in M, s \) has equal number of open and close parantheses.

Proof: By induction.

- Base case: \( \lambda \) has zero open and zero close parantheses.

- Constructor case: must show \( P(r) \) for \( r = (s)t \) assuming \( P(s) \) and \( P(t) \).
  
  - Let \( n_s, n_t \) be of open parantheses (= number close parantheses by hypothesis) in \( s, t \) respectively.
  
  - Then number of open parantheses in expression is \( n_s + n_t + 1 \).

- Similarly for close parantheses.