Pigeon-hole

Pigeon-hole principle: If $n$ objects are placed into $r$ boxes, then at least $\lceil n/r \rceil$ must be in the same box.

**Puzzle about white/black socks in a drawer, how many must you select before you have a matching pair? holes – colors, pigeons – socks.**

**Claim:** Five cities on an alien planet, at least 4 are in same hemisphere.

**Proof:** Draw great circle through two of five points, by pigeonhole two of remaining three cities are in same hemisphere.

**Claim:** Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set $A$ of size $n + 1$, at least two elts are relatively prime.

**Proof:**

- Pigeons – elts of $A$
- Holes – hole $i$ is for pigeons between $i$ and $i + 1$

That is, at least two numbers are 1 apart and hence relatively prime.

**Claim:** Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set $A$ of size $n + 1$, there are always two numbers such that one divides the other.

**Proof:** For each $a \in A$, write $a = 2^k m$, where $m$ is odd and $1 \leq m \leq 2n - 1$.

- Pigeons – elts of $A$
- Holes – odd numbers $m$

By pigeon-hole, must be two numbers $a$ and $a'$ with same hole $m$. That is, $a = 2^k m$ and $a' = 2^k m$, so one is a multiple of the other.

**Probability**

**Monty Hall Problem:**

- three doors, one car, two goats
- you pick a door
- host opens another door
- should you stay or switch?

**Assumptions:**

- prize is behind a random door
- you pick a random door
- host opens a door with a goat
- if host can pick door, picks a random door
Four Step Method

Step 1: Defining the sample space.

Def: *outcome* is any possible result of random choices.

Def: *sample space* is set of all outcomes.

Example: Tree diagram for Monty Hall, doors $A, B, C$

- root decides door for prize
- level one decides door you pick
- level two decides door host picks
- label leaves with path from root

Leaves are sample space

$\{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}$

Step 2: Defining events of interest.

Def: *event* is set of outcomes of interest

Example:

- prize is behind door $A$:
  $\{(A, A, B), (A, A, C), (A, B, C), (A, C, B)\}$

- you picked right door first time:
  $\{(A, A, B), (A, A, C), (B, B, A), (B, B, C), (C, C, A), (C, C, B)\}$

Step 3: Determine outcome probabilities

Modeling problem.

- assign edge probabilities
- compute outcome probabilities by product

Example: Compute outcome probabilities for Monty Hall

Step 4: Compute event probabilities

By summing outcomes in event.

Example: Compute $\Pr\{\text{switching wins}\}$ in Monty Hall

Probability Spaces

Def: countable sample space $S$ is non-empty countable set, $w \in S$ is outcome

Def: probability function $\Pr\{\} : S \to [0, 1]$ is s.t.

- $\forall w \in S, \Pr\{w\} \geq 0$
\[ \sum_{w \in S} \Pr\{w\} = 1 \]

**Def:** probability space is sample space plus probability function

**Def:** probability of event \( E \)

\[ \Pr\{E\} = \sum_{w \in E} \Pr\{w\} \]

**Implications**

**Claim:** (Sum rule) if \( E \cap F = \emptyset \), \( \Pr\{E \cup F\} = \Pr\{E\} + \Pr\{F\} \)

**Claim:** (Complement rule) \( \Pr\{A\} = 1 - \Pr\{A^c\} \) where \( A^c = \{w|w \notin A\} \)

**Claim:** (Union bound)

\[ \Pr\{E_1 \cup \ldots \cup E_n\} \leq \sum_{i=1}^{n} \Pr\{E_i\} \]

**Uniform Probability Space**

**Def:** probability space uniform if \( \Pr\{w\} \) same for all \( w \in S \)

**Claim:** for uniform prob., \( \Pr\{E\} = |E|/|S| \)

**Infinite Probability Space**

**Example:** coin flipping

- two players alternate flipping coin until comes up heads
- person who flips heads wins

Draw tree diagram.

- infinite tree

- outcome space \( T^nH \)

- probability function?

\[ \Pr\{T^nH\} = (1/2)^{n+1} \]

- valid? non-negative and

\[ \sum_{i=0}^{\infty} (1/2)^{n+1} = (1/2) \sum_{i=0}^{\infty} (1/2)^n = 1 \]

**Question:** prob. player 1 wins?

\[ \Pr\{1 \text{ wins}\} = \frac{1}{2} + \frac{1}{8} + \ldots = \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{2}{3} \]

**Question:** alternate solution?

- let \( p \) be prob. 1 wins, \( q \) be prob. 2 wins
- one of them wins: \( p + q = 1 \)
- 1 flip, then 2 is first player: \( p = (1/2) + (1/2)q \)
- solve eqns, \( p = 2/3 \)

**Conditional Probability**

**Example:** Prob. person \( p \)

- \( A \) = lives in Evanston?
- \( B \) = goes to Northwestern?
- \( B \) given \( A \)?
Draw Venn diagram.

**Def:** prob. $A$ given $B$,
\[
\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}
\]

**Claim:** (Product Rule)
\[
\Pr[A \cap B] = \Pr[A|B] \Pr[B] = \Pr[B|A] \Pr[A]
\]

**Claim:** (Product Rule)
\[
\Pr[E_1 \cap \ldots \cap E_n] = \\
\Pr[E_1] \Pr[E_2|E_1] \ldots \Pr[E_n|E_1 \cap \ldots \cap E_{n-1}]
\]

**Note:** Justifies tree diagram

- prob. on edge is conditional probability of edge given reach parent node
- multiple to get outcome prob. by above defn.

**Example:** A family has two children and one is a boy. What is probability other is a boy?

- $\Omega = \{BB, BG, GB, GG\}$
- $A = \{BB, BG, GB\}$
- $B = \{BB\}$
- $P(A) = 3/4$
- $P(A \cap B) = P(\{BB\}) = 1/4$
- $P(B|A) = 1/3$

**Claim:** Law of alternatives: If $A_1, \ldots, A_n$ are disjoint events whose union is $S$, then
\[
P(B) = \sum_{i=1}^{n} \Pr(A_i) \Pr(B|A_i)
\]

**Claim:** Bayes Thm: \( P(A|B) = P(B|A)P(A)/P(B) \)

**Proof:** \( P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) \)

**Example:** Cards: blue/blue, blue/red, red/red.

**Question:** Random side of random card is red, what is prob. other side is red?

- card 1: two blue, label sides front/back
- card 2: two red, front/back
- card 3: red side front, blue side back
- \( S = \{(i, x)\} \) where $i$ is card, $x$ is front or back
- $A =$ red side shown $= \{2F, 2B, 3F\}$, $P(A) = 1/2$
- $B =$ card 2 chosen, $= \{2F, 2B\}$
- $P(B|A) = P(A \cap B)/P(A) = (1/3)/(1/2) = 2/3$