Counting

Bijection Rule

Def: bijection \( f : A \rightarrow B \) is perfect matching of \( a \in A \) and \( b \in B \)

Claim: if exists bijection \( f : A \rightarrow B \), then \( |A| = |B| \)

Example: doughnuts

- \( A = \) ways to select dozen doughnuts from 5 varieties
- \( B = \) number 16-bit strings with exactly 4 ones

representation:

- element of \( A \):
  - 2 choc., 0 lemon, 5 sugar, 2 glazed, 2 plain
    - 00 − ... − 00000 − 00 − 00
  - choc.—lemon—sugar—glazed—plain

- element of \( B \):
  - replace “−” with 1

bijection:

- dozen doughnuts
  - \( c \) choc., \( l \) lemon, \( s \) sugar, \( g \) glazed, \( p \) plain

- bit string
  - \( c \) 0’s, \( l \) 0’s, \( s \) 0’s, \( g \) 0’s, \( p \) 0’s

[How to count bit-strings? or other sequences?]

Sum/Product Rule

\( A = \) set of cakes

\( B = \) set of pies

Question: How many ways are there to pick one cake and one pie?

\( |A| \times |B| \)

Question: How many ways to pick one dessert?

\( |A| + |B| \)

Question: How many ways are there to pick one cake and one pie if some cakes are pies?

[Boston creme pie]

Draw Venn diagram – \( |A \cup B| = |A| + |B| - |A \cap B| \)

In general, given finite sets \( A_1, \ldots, A_n \),

- Product Rule: There are \( \prod_{i=1}^{n} |A_i| \) ways to select \( n \) elements, one element from each set.

- Inclusion/Exclusion: There are \( | \cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} |A_i \cap A_j| + \ldots + \)
\((-1)^{n+1}|A_1 \cap \ldots \cap A_n|\) ways to select one element.

**Example:** Sum/Product Rule:

- How many two-digit numbers are there?
  9 choices for 1st digit \(\times\) 10 for 2nd = 90
- If \(|A| = n\) and \(|B| = m\), how many functions are there from \(A\) to \(B\)?
  \(n\) choices for 1st elt \(\times\) \(\ldots\) \(\times\) \(n\) for \(m\)'th elt = \(m^n\)
- How many subsets of an \(n\)-element set?
  bijection to binary strings: subset \(S\) maps to string \(s\) with \(i\)'th bit 1 iff \(i \in S\)
  2 choices for each of \(n\) bits = \(2^n\)
- How many passwords consisting of letters and digits that
  - start with a letter
  - have length 6-8

Let \(F\) be set of first symbol, \(S_5, S_6, S_7\) be strings of 5, 6, 7 symbols

Password is one elt of \(F\) appended with one elt of \(S_5\) or \(S_6\) or \(S_7\)

\[
(F \times S_5) \cup (F \times S_6) \cup (F \times S_7)
= |F \times S_5| + |F \times S_6| + |F \times S_7|
= |F| \cdot |S_5| + |F| \cdot |S_6| + |F| \cdot |S_7|
= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7
1.8 \times 10^{14}
\]

**Example:** Inclusion/Exclusion, Sieve of Eratosthenes (200 B.C.)

How many primes < 100?

- Count composites: have a prime divisor \(\leq 10\).
- Primes \(\leq 10\) are 2, 3, 5, 7.
- Let
  - \(A_2 = \{n \leq 100 : 2|n\}\)
  - \(A_3 = \{n \leq 100 : 3|n\}\)
  - \(A_5 = \{n \leq 100 : 5|n\}\)
  - \(A_7 = \{n \leq 100 : 7|n\}\)
- \# composites = \(|A_2 \cup A_3 \cup A_5 \cup A_7| - 4\)
  (for primes 2, 3, 5, 7)
- \(|A_p| = \lfloor \frac{100}{p} \rfloor\), \(|A_{p,q}| = \lfloor \frac{100}{pq} \rfloor\), etc.
  \(|A_2 \cup A_3 \cup A_5 \cup A_7| = \ldots = 78\)
- \# primes = 99 - (78 - 4) = 99 - 74 = 25.

Sieve crosses out numbers divisible by 2 and thereafter numbers divisible by first on list.

**Permutations**

Generalized product rule: sequences of \(r\) elts of \(n\)-elt set is

\(n(n-1)(n-2)\ldots(n-r+1)\)

**Example:** Race with \(n\) horses, how many options for win, place, show?

By product rule, \(n(n-1)(n-2)\).

**Example:** Chess problem

\# ways place white rook, black rook, neither attacks other

- \((c_w, r_w) = \text{col/row of white rook}\)
- \((c_b, r_b) = \text{col/row of black rook}\)
• board position is 4-digit sequence 
  \((c_w, r_w, c_b, r_b)\)
  e.g., (1, 1, 8, 8) or (8, 8, 1, 1)
• \(c_w, r_w\) have 8 choices each
• given \((c_w, r_w)\), \(c_b, r_b\) have 7 choices each

is 56^2.

**Def:** permutation \(\sigma(n)\) of \(n\) distinct objects is an ordered selection.

**Claim:** \# permutations of \(n\) elements is

\[ n! = n(n - 1) \ldots 1. \]

[[proof - product rule]]

### Division Rule

**Claim:** If \(f: A \to B\) maps exactly \(k\) els of \(A\) to each elt of \(B\), then \(|A| = k|B|\).

**Example:** Chess problem

\# ways place two white rooks in diff row/col

• \((c_1, r_2) = \text{col/row of first rook}\)
• \((c_2, r_2) = \text{col/row of second rook}\)
• board position is 4-digit sequence 
  \((c_1, r_1, c_2, r_2)\)
  but \((1, 1, 8, 8) = (8, 8, 1, 1)\)
• so divide by 2

is \(56^2/2\).

### Combinations

**Example:** pizza toppings

\# pizzas with 2 toppings

• pepper
• onion
• mushroom

**Product+division rule:**

• \((t_1, t_2)\) sequence of toppings
• \((t_1, t_2) = (t_2, t_1)\) so divide by 2
• \(3 \times 2 = 6\) sequences
• so 3 ways

**Def:** combination is unordered selection of \(r\) objects from \(n\) objects

**Claim:** \# combinations of \(r\) from \(n\) is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}.
\]

**Proof:** bijection

• take permutation of \(n\)
• let first \(r\) be selection
• fix permutation:
  - any other permutation with same first \(r\) gives same unordered selection
  - order of first \(r\) and last \(n-r\) doesn’t matter
  - \(r!\) perms of first \(r\), \((n-r)!\) perms of rest
• \(r!(n-r)!\) ways to unorder them

**Example:** \# \(n\)-bit seq with exactly \(k\) ones

[[recall came up in doughnut problem]]
Example: # ternary strings with $k_1$ 1’s and $k_2$ 2’s?

**Example:** Rearrangements of BOOK-KEEPER

- pretend distinct: 10! ways
- fix over-counting: divide by
  - 2! (switch O’s)
  - 2! (switch K’s)
  - 3! (reorder E’s)

Example: # walks 5 blocks in each direction

- sequences of 5 N’s, 5 W’s, 5 E’s, 5 S’s
- so: 20!/(5!)^4

Bars and Stars

**Important that all elements in set and in selection are distinct. Finding permutations/combinations with repetition is slightly harder.**

Question: How many ways to give $n$ candycorn to $k$ kids so each kid gets at least one piece of candy?

- Stars = candy, draw on line
- Bars = bucket boundaries, draw between stars
- $n - 1$ places for bucket boundaries; $k - 1$ boundaries
- Answer: $\binom{n-1}{k-1}$

Question: How many ways if some kids can get no candy?

**Claim:** Answer: $\binom{n+k-1}{k-1}$

**Proof:**

1. Placing $n$ objects in $k$ bins allowing empty bins is like placing $n + k$ objects in $k$ bins disallowing empty bins
2. Have $n + k - 1$ symbols of which $k - 1$ must be bars

Note above two are consistent; to give each kid at least one candycorn, could have taken $k$ candycorn out of the $n$ and then done the second approach. Get same answer.

Example:

- How many positive solutions to $a + b = 10$?
- How many non-negative solutions to $a + b = 10$?

**Identities**

What is $\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r}$?

Often have a nice combinatorial proof. Some easy ones:

- $\binom{n}{r} = \binom{n}{n-r}$: choose elements to keep or to throw away.
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$: all subsets of an $n$-element set.
- $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$ (the binomial theorem): coeff of $x^{n-j} y^j$, must choose $j$ terms in product from which to take $y$. 
\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \] (Pascal’s triangle): remove arbitrary element, either this is in the subset of size \( k \) (first term) or not (second term).

Claim: \( \sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n} \)

Proof: Let \( S \) be deck of cards with \( n \) red cards and \( 2n \) black cards.

- RHS: \( \binom{3n}{n} \) is the number of \( n \)-card hands.
- LHS:
  - \( \binom{n}{r} \) is the number of way to pick \( r \) red cards
  - \( \binom{2n}{n-r} \) is the number of ways to pick \( n-r \) black cards

so \( \binom{n}{r} \binom{2n}{n-r} \) is number of ways to pick a hand with exactly \( r \) red cards.

Result follows by summing over \( r \). \( \square \)

\section*{Poker Hands}

52 cards in a deck:

- 4 suits: Spades, Hearts, Diamonds, Clubs
- 13 values: 1,\ldots, 10, Jack, Queen, King, Ace

5-card draw: each player given 5 cards

\( \binom{52}{5} = 2,598,960 \) possible hands.

How many ways to get:

- Four-of-a-kind – 4 cards with same value
  \( \{2H, 2S, 2D, 2C, 5S\} \)

Hand fully specified by sequence specifying
  - value of 4 cards
  - value of extra card
  - suit of extra card

Count ways to pick sequence
  - 13 choices for value of 4 cards
  - 12 choices for value of extra card
  - 4 choices for suit of extra card
so \( 13 \cdot 12 \cdot 4 = 624 \) four-of-a-kinds, or about 1 in 4000 hands.

- Full house – 3 cards of one value, 2 cards of another value
  \( \{7H, 7S, 7D, JH, JS\} \)

Sequence
  - value of triple – 13
  - suits of triple – \( \binom{4}{3} \)
  - value of pair – 12
  - suits of pair – \( \binom{4}{2} \)
so \( 13 \cdot 4 \cdot 12 \cdot 6 = 3744 \), or about 1 in 700 hands.

- Two pair – 2 cards of one value, 2 cards of another value
  \( \{4S, 4H, 6D, 6H, KC\} \)

Sequence
  - value of first pair – 13
– suits of first pair – \( \binom{4}{2} \)
– value of second pair – 12
– suits of second pair – \( \binom{4}{2} \)
– value of extra card – 11
– suit of extra card – 4

\[ \text{Wrong Answer!} \]

Each hand gives rise to two distinct sequences:

\[ (4, \{ S, H \}, 6, \{ D, H \}, K, C) \]
\[ = (6, \{ D, H \}, 4, \{ S, H \}, K, C) \]

Solution: mapping is 2-to-1, so divide by 2 (division rule)

Number of two-pair hands is: \( 13 \cdot 6 \cdot 12 \cdot 6 \cdot 11 \cdot 4/2 = 123,552 \), or about 1 in 20.

Note: alternatively, could come up with different sequence to count, e.g.,

– values of two pairs – \( \binom{13}{2} \)
– suits of lower-valued pair – \( \binom{4}{2} \)
– suits of higher-valued pair – \( \binom{4}{2} \)
– value of extra card – 11
– suit of extra card – 4

- Hands with every suit
  \{ AS, AH, JD, 4C, 10H \}

Sequence

– value of spade – 13
– value of heart – 13
– value of diamond – 13
– value of club – 13
– suit of extra card – 4
– value of extra card – 12

Overcount?

\[ (A, A, J, 4, H, 10) = (A, 10, J, 4, H, A) \]

Must divide by 2.

**Pigeon-hole**

Pigeon-hole principle: If \( n \) objects are placed into \( r \) boxes, then at least \( \lceil n/r \rceil \) must be in the same box.

**Claim:** Consider numbers \( \{ 1, 2, \ldots, 2n \} \).

Then in any set \( A \) of size \( n + 1 \), at least two els are relatively prime.

**Proof:**

- Pigeons – els of \( A \)
- Holes – hole \( i \) is for pigeons between \( i \) and \( i + 1 \)

That is, at least two numbers are 1 apart and hence relatively prime. \( \square \)

**Claim:** Consider numbers \( \{ 1, 2, \ldots, 2n \} \).

Then in any set \( A \) of size \( n + 1 \), there are always two numbers such that one divides the other.

**Proof:** For each \( a \in A \), write \( a = 2^k m \), where \( m \) is odd and \( 1 \leq m \leq 2n - 1 \).

- Pigeons – els of \( A \)
- Holes – odd numbers \( m \)

By pigeon-hole, must be two numbers \( a \) and \( a' \) with same hole \( m \). That is, \( a = 2^k m \) and \( a' = 2^{k'} m \), so one is a multiple of the other. \( \square \)

**Claim:** Five cities on an alien planet, at least 4 are in same hemisphere.


Proof: Draw great circle through two of five points, by pigeonhole two of remaining three cities are in same hemisphere.

Magic Trick

\[\text{[Ok to skip.]}\]

- Audience: choose 5 cards
- Assistant: show magician 4 cards, one at a time
- Magician: announce missing card

Counting:

- Audience: \(\binom{52}{5}\)
- Assistant: \(4! = 24\) ways to show 4 of five cards

but ... 48 possible cards, so not enough permutations to narrow it down.

Idea: Assistant gets to pick order and also which card to leave out!

Create bipartite graph, find matching:

- LHS: all audience choices
- RHS: all sequences of 4 distinct cards
- edges: if sequence valid for audience choice

Recall: Claim: There is a matching from LHS to RHS if each subset of LHS has neighbor set of larger cardinality.

- LHS: \(\binom{5}{4!} = 120\) edges from a LHS vertex
- RHS: 48 edges from a RHS vertex

Consider subset \(S\) of LHS.

- \(120|S|\) edges leave subset
- \(48|N(S)|\) enter neighbor set

so \(|N(S)| > |S|\), and there is a matching by Hall’s Theorem.

Real trick:

- At least 2 cards have same suit, one of which is at most 6 hops clockwise from the other in a cycle.
- Hide clockwise card; reveal other card first.
- Use remaining three cards to indicate number of hops, encoding 1, \ldots, 6 using total order (e.g., low-medium-high or low-high-medium).