# Minimum Face-Spanning Subgraphs of Plane Graphs
## (Extended Abstract)

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**Abstract.** A plane graph is a planar graph with a fixed embedding. Let $G = (V, E)$ be an edge weighted connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively. Let $F$ be the set of faces of $G$. For each edge $e \in E$, let $w(e) \geq 0$ be the weight of the edge $e$ of $G$. A face-spanning subgraph of $G$ is a connected subgraph $H$ induced by a set of edges $S \subseteq E$ such that the vertex set of $H$ contains at least one vertex from the boundary of each face $f \in F$ of $G$. A minimum face-spanning subgraph $H$ of $G$ is a face-spanning subgraph of $G$, where $\sum_{e \in S} w(e)$ is minimum. In this paper we consider the problem of finding a minimum face-spanning subgraph of a plane graph and deal with the following problem which we call "the face-spanning subgraph problem": "Is there any face-spanning subgraph $H$ of $G$ such that $\sum_{e \in S} w(e) \leq b$, for a positive real number $b$?". We prove that the face-spanning subgraph problem of a plane graph is $NP$-complete, which implies that it is unlikely to have a polynomial time algorithm for finding a minimum face-spanning subgraph of a plane graph. In this paper, we prove a variation of the face-spanning subgraph problem called "minimum-vertex face-spanning subgraph problem" is also $NP$-complete. We also present approximation algorithms for both the problems.

**Key words:** Graphs, $NP$-complete problems, Connected vertex cover, Face-spanning subgraphs, Weighted tree cover.

## 1 Introduction

A gas company wants to supply gas to a locality from a single gas source. They are allowed to pass the underground gas lines along the road network only, because no one allows to pass gas lines through the bottom of his building. The road network divides the locality into many regions as illustrated in Figure 1(a), where each road is represented by a line segment and a point at which two or

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more roads meet is represented by a (black or white) small circle. A point at which two or more roads meet is called an intersection point. Each region is bounded by some line segments and intersection points. These regions need to be supplied gas. If a gas line reaches an intersection point on the boundary of a region, then the region may receive gas from the line at that intersection point. Thus the gas lines should reach the boundaries of all the regions of the locality. Gas will be supplied from a gasfield which is located outside of the locality and a single pipe line will be used to supply gas from the gasfield to an intersection point on the outer boundary of the locality. The gas company wants to minimize the establishment cost of gas lines by selecting the roads for laying gas lines such that the total length of the selected roads is minimum. Since gas will be supplied from the gasfield using a single line to the locality, the selected road network should be connected and contains an intersection point on the outer boundary of the locality. Thus the gas company needs to find a set of roads that induces a connected road network, supply gas in all the regions of the locality and the length of the induced road network is minimum. Such a set of roads is illustrated by thick lines in Figure 1(b).

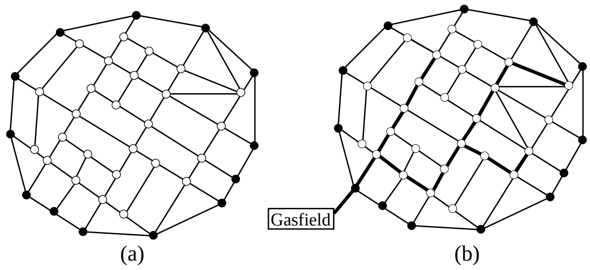


**Fig. 1.** (a) A road-network of a locality and (b) a sample setup of gas pipelines drawn by thick lines for supplying gas in all the regions from a gasfield.

The problem mentioned above can be modeled using a plane graph as follows. Let $G = (V, E)$ be an edge weighted connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively. Let $F$ be the set of faces of graph $G$. For each edge $e \in E$, $w(e) \geq 0$ is the weight of the edge $e$ of $G$. A face-spanning subgraph of $G$ is a connected subgraph $H$ induced by a set of edges $S \subseteq E$ such that the vertex set of $H$ contains at least one vertex from the boundary of each face $f \in F$ of $G$. Figure 2 shows two face-spanning subgraphs drawn by thick lines where the cost of the face-spanning subgraph in Figure 2(a) is 11 and the cost of the face-spanning subgraph in Figure 2(b) is 13. Thus a plane graph may have many face-spanning subgraphs whose cost are different. A minimum face-spanning subgraph $H$ of $G$ is a face-spanning subgraph of $G$, where $\sum_{e \in S} w(e)$ is minimum, and a minimum face-spanning subgraph problem asks to find a minimum face-spanning subgraph of a plane graph. If we represent each road of the road network by an edge of $G$, each intersection point by a vertex of $G$, each region by a face of $G$ and assign the length of a road to the weight of the corresponding edge, then the problem of finding a minimum face-spanning subgraph of $G$ is the same as the problem of the gas company mentioned above. A minimum face-spanning subgraph problem often arises in applications like establishing power transmission lines in a city, power wires layout in a complex circuit, planning irrigation canal networks for irrigation systems etc.

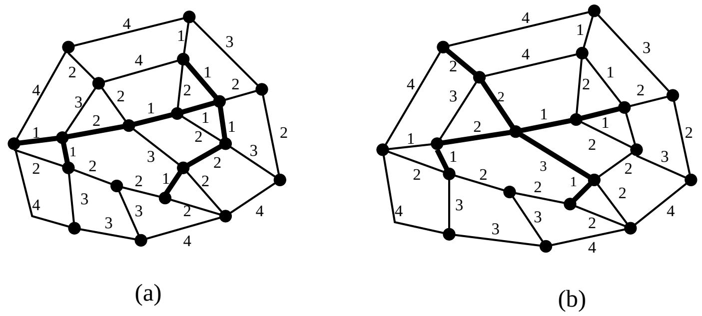


**Fig. 2.** A simple graph with (a) a face-spanning subgraph of cost 11 and (b) another face-spanning subgraph of cost 13.

One may think of the "vertex cover problem" [FD04] or the "face cover problem" [AL04] while considering the minimum face-spanning subgraph problem. Unfortunately, the minimum face-spanning subgraph problem is quite different from those two problems. A vertex set $C \subseteq V$ is called a *vertex cover* if every edge of $G$ is incident to some vertex in $C$ and the *vertex cover problem* asks to compute a minimum vertex cover in given $G$. The vertex cover problem asks to find a vertex set which contains at least one vertex from the end vertices of each edge of $G$ whereas the minimum face-spanning subgraph problem asks to find an edge set which contains at least one vertex from the boundaries of each face of $G$. Hence, the vertex cover problem and the minimum face-spanning subgraph problem are different. A set of faces whose boundaries contain all vertices in a plane graph $G$ is said to be a *face cover* for $G$ and the *face cover problem* asks to compute a minimum face cover in given $G$. The face cover problem asks to find a face set which contains all the vertices of $G$ whereas the minimum face-spanning subgraph problem asks to find an edge set which contains at least one vertex from the boundaries of each face of $G$. Hence, the face cover problem and the minimum face-spanning subgraph problem are also different.

Efficient algorithms are necessary to solve these kinds of problems, which arise from numerous practical applications. However developing efficient algorithms is not always possible for many such problems [AHU74,K72,GJ79]. In this paper, we show that it is unlikely to have a polynomial-time algorithm for finding a minimum face-spanning subgraph of a plane graph by proving that the decisive version of the minimum face-spanning subgraph problem belongs to the infamous class of *NP*-complete problems. In such a case, design of approximation algorithms is needed for practical applications. We thus present approximation algorithms for both the problems in this paper.

The rest of the paper is organized as follows. Section 2 describes some definitions. Section 3 proves the *NP*-completeness of the face-spanning subgraph problem. A variation of this problem called the minimum-vertex face-spanning subgraph problem is discussed in Section 4. We deal with approximation algorithms in Section 5. Finally Section 6 gives the conclusion.

## 2 Preliminaries

In this section we give some definitions.

Let $G = (V, E)$ be a simple graph with vertex set $V$ and edge set $E$. The number of vertices of $G$ is denoted by $n$, that is, $n = |V|$, and the number of edges of $G$ is denoted by $m$, that is, $m = |E|$. We often denote the set of vertices of $G$ by $V(G)$ and the set of edges of $G$ by $E(G)$. We denote an edge joining vertices $v_i, v_j$ of $G$ by $(v_i, v_j)$. If $(v_i, v_j) \in E$, then two vertices $v_i, v_j$ are said to be *adjacent* in $G$; edge $(v_i, v_j)$ is then said to be *incident* to vertices $v_i$ and $v_j$; $v_i$ is a *neighbor* of $v_j$. The *degree* of a vertex $v$ in $G$ is the number of edges incident to $v$ in $G$. We denote the maximum degree of graph $G$ by $\Delta(G)$ or simply by $\Delta$. A *path* in $G$ is an ordered list of distinct vertices $(v_1, v_2, \ldots, v_{q-1}, v_q) \in V$ such that $(v_{i-1}, v_i) \in E$ for all $2 \leq i \leq q$. $G$ is *connected* if for any two distinct vertices $v_i$, $v_j$ of $G$ there is path between $v_i$ and $v_j$ in $G$. $G$ is a *tree* if $G$ is connected and has no cycle. $H = (V', E')$ is called a *subgraph* of $G$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph $H = (V', E')$ of $G$ is called the *edge induced subgraph* of $G$ induced by the edge set $E'$ if $V'$ contains only the vertices of $G$ which are end vertices of the edges in $E'$. For a set of edges $S \subseteq E$, we denote by $V(S)$ the set of vertices consisting of the end vertices of the edges in $S$, that means, $V(S)$ is the set of vertices of the edge induced subgraph of $G$ induced by $S$.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* $G$ is a planar graph with a fixed embedding. A plane graph divides the plane into connected regions called *faces*. The unbounded region is called the *outer face* of $G$. We call a vertex $v$ of $G$ an *outer vertex* if it is on the boundary of the outer face of $G$. Let $F$ be the set of faces of plane graph $G$. We say a set of edges $S \subseteq E$ *covers a face* $f \in F$ if $V(S)$ contains at least one vertex from the vertices on the boundary of $f$. *Edge weighted connected plane graph* is a connected plane graph where each edge $e$ has a weight $w(e) \geq 0$. Let $G$ be an edge weighted connected plane graph. Then for a subgraph $H$ of $G$, the *cost of* $H$ is computed as $\sum_{e \in E(H)} w(e)$. We say a vertex set $V' \subseteq V$ *covers all the edges* of $G$ if $V'$ contains at least one vertex from the end vertices of each edge of $G$. $H$ is a *tree cover* of $G$ if $H$ is a tree in $G$ and the vertices in $H$ cover all the edges of $G$. A *face-spanning subgraph* of edge weighted connected plane graph $G$ is a connected subgraph $H$ induced by a set of edges $S \subseteq E$ such that the vertex set of $H$ contains at least one vertex from the boundary of each face $f \in F$ of $G$. A minimum face-spanning subgraph $H$ of $G$ is a face-spanning subgraph of $G$, where $\sum_{e \in S} w(e)$ is minimum.

To prove that the face-spanning subgraph problem is *NP*-complete we reduce the weighted tree cover problem to the face-spanning subgraph problem in Section 3. The formal definition of weighted tree cover problem is as follows [AHH93]:

**Definition 1.** (Weighted Tree Cover Problem) *Given a plane graph $G = (V, E)$ and weight on the edges $w(e) \geq 0$ for all $e \in E$ and a positive real number $b$, does there exist any weighted tree $T = (V', E')$ induced by $E' \subseteq E$, whose vertices $V' \subseteq V$ cover all the edges in $E$ and $\sum_{e \in E'} w(e) \leq b$?*

## 3 Face-Spanning Subgraph Problem

In this section we show that the face-spanning subgraph problem is *NP*-complete. To prove that the face-spanning subgraph problem is *NP*-complete, we have to show that (i) the face-spanning subgraph problem is in *NP* and (ii) the face-spanning subgraph problem is *NP*-hard. We begin with a formal definition of the face-spanning subgraph problem:

**Definition 2.** (Face-Spanning Subgraph Problem) *Let $G = (V, E)$ be a connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively, and let $F$ be the set of faces of graph $G$. Let $w(e) \geq 0$ be a positive real number assigned to edge $e$ as weight for every edge $e \in E$. Then is there any set $S \subseteq E$ such that the subgraph $H$ induced by $S$ is connected, cover all faces of $G$ and the cost of $H$ is $\leq b$, for a given positive real number $b$?*

In the rest of this section we prove that the face-spanning subgraph problem defined above is *NP*-complete. As the first step of the proof, we prove the following lemma.

**Lemma 1.** *The face-spanning subgraph problem is in NP.*

*Proof.* To prove that the face-spanning subgraph problem is in *NP*, it is sufficient to prove that for a given set $S \subseteq E$, we can verify in polynomial-time that the subgraph $H$ induced by $S$ (i) is connected, (ii) cover all faces of $G$ and (iii) the cost of $H$ is $\leq b$.

**(i)** Connectivity of the subgraph $H$ induced by $S$ can be checked using DFS in linear time.

**(ii)** We can verify whether $S$ covers all faces of $G$ or not in linear time by the following method.

Let $F(v)$ be the set of faces of $G$ such that each face in $F(v)$ contains the vertex $v$. We maintain a face-list for each vertex $v$, where the face-list for $v$

contains the faces in $F(v)$. We also maintain a boolean array $AF$ of length $|F|$ to indicate whether the faces of $G$ are covered by the vertices in $V(S)$ or not. For all $j \in \{1, 2, \ldots, |F|\}$, $AF[j]$ corresponds to the face $f_j$ of graph $G$. Initially all elements of $AF$ are set to 0 to indicate that no face is covered by the vertices in $V(S)$ initially. We traverse the face-list for each vertex $v$ in $V(S)$ and for each face $f_j$ in the face-list, we change the value of $AF[j]$ to 1 to indicate that the face $f_j$ is covered by the vertices in $V(S)$. After traversing the face-lists for all vertex in $V(S)$, we check the array $AF$ to know that whether all faces of $G$ are covered or not.

We now calculate the complexity of the method described above. Since $|F(v)|$ is equal to the degree of $v$ and $|V(S)|$ is at most $|V|$, then, to check all the vertices of $V(S)$, we have to consider at most $\sum_{v \in V(S)} d(v) = 2m = O(m) = O(n)$ entries in total. Since the length of array $AF$ is equal to $|F|$, the traversing time of $AF$ is $O(|F|) = O(n)$. Thus, the overall time complexity to verify that whether the vertices in $V(S)$ cover all faces of $G$ or not is $O(n)$.

**(iii)** It can be verified easily in $O(n)$ time that the cost of $H$ is $\leq b$.

Since it is possible to verify (i), (ii) and (iii) in polynomial time, the face-spanning subgraph problem is in *NP*. $\mathcal{Q.E.D.}$

We now prove the following lemma as the second part of the proof.

**Lemma 2.** *The face-spanning subgraph problem is NP-hard.*

To prove lemma 2 we will prove that, the *NP*-complete problem weighted tree cover problem can be polynomially transformed into the face-spanning subgraph problem in such a way that any polynomial-time algorithm for solving the face-spanning subgraph problem could be used to solve the weighted tree cover problem in polynomial time.

Let $G = (V, E)$ be a connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively. Let $F$ be the set of faces of graph $G$. We obtain a graph $G'$ from $G$ as follows. For each edge $e = (v_k, v_l) \in E$ we add a vertex $v_e$ and two edges $(v_k, v_e)$ and $(v_l, v_e)$ to $G$. More formally, $G' = (V', E')$ where $V' = V \cup V_e$, $V_e = \{v_e | e \in E\}$ and $E' = E \cup E_e$ where $E_e = \{(v_e, v_k), (v_e, v_l) | \{e = (v_k, v_l)\} \in E\}$. In $G'$ we call a vertex in $V_e$ a *new vertex*, a vertex in $V$ an *original vertex*, an edge in $E_e$ a *new edge* and an edge in $E$ an *original edge*. Note that original vertices and original edges of $G'$ are also the vertices and edges of $G$. We assign the cost $w(e)/2$ to each of the edges $(v_e, v_k)$ and $(v_e, v_l)$ for all $e = (v_k, v_l) \in E$. Figure 3 illustrates the construction of $G'$ where the vertices drawn by white small circles are new vertices, the edges drawn by dashed lines are new edges, the vertices drawn by black circles are original vertices and the edges drawn by solid lines are the original edges of $G'$. If $G$ has $n$ vertices and $m$ edges, then $G'$ has $n + m$ vertices and $3m$ edges. Clearly $G'$ can be constructed from $G$ in $O(n)$ time. One can easily observe that the graph $G'$ is planar as illustrated in Figure 3, where a plane embedding of $G'$ is shown. Throughout the paper we consider $G'$ as a plane embedding of the graph $G'$. For each edge $e = (v_k, v_l) \in E$, we call the face $(v_k, v_l, v_e)$ of $G'$ a $\alpha$-face. We call each of the remaining faces of $G'$ a $\beta$-face. Figure 3 illustrates $\alpha$-faces and $\beta$-faces. We now have the following lemma.

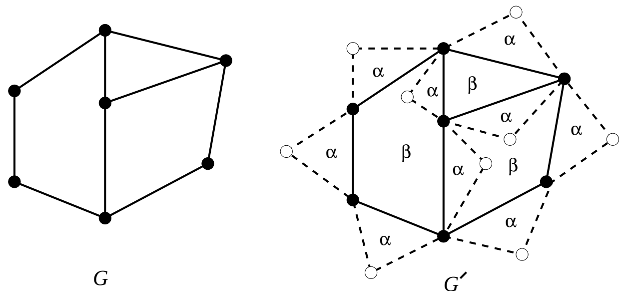


**Fig. 3.** Illustration for the construction of $G'$ from $G$.

**Lemma 3.** *$G'$ has a face-spanning subgraph $H$ of cost $\leq b'$ if and only if $G$ has a weighted tree cover $T$ of cost $\leq b$, where $b$ and $b'$ are two positive real numbers.*

*Proof.* **Necessity.** Assume that $G'$ has a face-spanning subgraph of cost $\leq b'$, that is, there is an edge set $S' \subseteq E'$ of graph $G'$ such that the subgraph $H$ induced by $S'$ is connected, cover all faces of $G'$ and the cost of $H$ is $\leq b'$. We now prove that $G$ has a weighted tree cover $T$ of cost $\leq b$, for a positive real number $b$.

From the construction of $G'$ it is obvious that the degree of each new vertex $v$ is two in $G'$. Each new vertex has exactly two neighbors $v_i$, $v_j$ among the original vertices and there is an original edge $(v_i, v_j)$ between the two original vertices as illustrated in Figure 4(a). Modifying the subgraph $H$ we construct a subgraph $T$ of $G'$ such that $T$ contains only the original vertices and original edges as follows. Since $H$ is a subgraph of $G'$, degree of each new vertex $v$ in $H$ is either one or two. For each new vertex $v$ of $H$ we perform one of the two operations described in Case 1 and Case 2 below to obtain $T$ from $H$.

*Case* 1: $v$ has degree two in $H$
In this case $v$ has two neighbors $v_i$, $v_j$ among the original vertices such that $(v_i, v_j)$ is a solid edge. If $(v_i, v_j) \in E(H)$ then we delete $v$ from $H$ to obtain $T$ as illustrated in Figure 4(b) and 4(e). Otherwise we replace the path $(v_i, v, v_j)$ of $H$ by the edge $(v_i, v_j)$ to construct $T$ as illustrated in Figure 4(c) and 4(e).

*Case* 2: $v$ has degree one in $H$
In this case $v$ has exactly one neighbor $v_i$ among the original vertices. We simply remove the new vertex $v$ of $H$ to construct $T$. Figure 4(d) and 4(f) illustrates this case.

If $T$ contains cycles, we delete an edge from each cycle until the resulting subgraph has no cycle and we regard the resulting subgraph as $T$, and take the set of all edges in $T$ as $S$.

We now prove that $T$ is a tree in $G$ of cost $\leq b$. Since $H$ is connected, if we delete the new vertex $v$ or we replace the path $(v_i, v, v_j)$ by edge $(v_i, v_j)$ in Case 1, $T$ remains connected. Again, the cost of the path $(v_i, v, v_j)$ is $w(e)/2 + w(e)/2 = w(e)$ in total, which is equal to the cost of the edge $(v_i, v_j)$. Hence in Case 1, the cost of the modified subgraph is decreased (if we delete the new vertex $v$) or unchanged (if the path $(v_i, v, v_j)$ is replaced by edge $(v_i, v_j)$). In Case 2, the new vertex has degree one and it is omitted, hence $T$ remains connected after considering Case 2 for all such new vertices. In this case, edge $(v, v_i)$ is removed, hence the cost of the modified subgraph decreases. Thus $T$ is a connected subgraph of cost $\leq b'$ in $G'$. Note that we have destroyed cycles to construct $T$ and $T$ is a tree of cost $\leq b'$ in $G'$. If we take $b = b'$, then the cost of tree $T$ is $\leq b$. Since the edges of $T$ in $G'$ are original edges and the vertices of $T$ in $G'$ are original vertices, $G$ contains $T$. Hence $T$ is a tree of cost $\leq b$ in $G$.

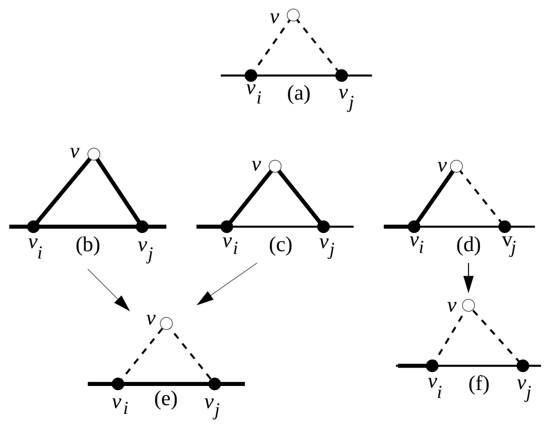


**Fig. 4.** Illustration for the construction of $T$ from $H$.

Note that $T$ and $H$ are induced by $S$ and $S'$ respectively. To prove that $T$ is a weighted tree cover in $G$ of cost $\leq b$, it is now remained to show that the set of vertices $V(S)$ of subgraph $T$ is a vertex cover in $G$. Since $H$ is face-spanning subgraph of $G'$, the set of vertices $V(S')$ of $H$ covers all faces of $G'$. Hence $V(S')$ contains at least one vertex (either black or white) from the boundary of each face of $G'$. Since $H$ is connected, $V(S')$ can contain a new vertex $v$ only if $V(S')$ contains at least one neighbor $v_i$ of $v$ among the original vertices in $G'$. Since a new vertex $v$ has degree 2, the two faces covered by a new vertex $v$ are also be covered by an original vertex $v_i$ which is neighbor to the new vertex $v$. Thus $V(S')$ contains at least one original vertex from the boundary of each face of $G'$. Since $V(S)$ contains all the original vertices of $V(S')$, $V(S)$ also contains at least one original vertex from the boundary of each face of $G'$. Since we create an $\alpha$-face in $G'$ for each edge of $G$ while constructing $G'$, there is a face of $G'$

for each edge in $G$. Since each face of $G'$ is covered by $V(S)$, each edge of $G$ is covered by $V(S)$. Hence $V(S)$ contains at least one vertex from each edge of $G$. Thus $V(S)$ is a vertex cover of $G$.

Since, $T$ is a tree in $G$ of cost $\leq b$ and $V(S)$ is a vertex cover of graph $G$, $T$ is a weighted tree cover of cost $\leq b$ of $G$.

**Sufficiency.** Assume that $G$ has a weighted tree cover of cost$\leq b$, that is, there is tree $T$ in $G$ of cost $\leq b$ and the vertex set $V(S)$ that $T$ contains is a vertex cover of graph $G$. We now prove that $G'$ has a face-spanning subgraph $H$ of cost $\leq b'$, for a positive real number $b'$. We take $b = b'$.

From the construction of $G'$ it is clear that all the vertices in $V(S)$ and the edges of $T$ in $G$ are also in $G'$. We take $S'$ as the set of edges of $G'$ which are in $S$ and let $H$ be the subgraph induced by $S$. Then $H$ contains all the edges of $T$ and $V(S')$ contains all the vertices in $V(S)$. We now show that the subgraph $H$ of $G'$ is (i) connected, (ii) cover all faces of $G'$ and (iii) the cost of $H$ is $\leq b'$.

**(i)** From the construction it is obvious that all the vertices and edges of $G$ are also in $G'$. Since $T$ is a tree in $G$ and $T = H$, $H$ is a tree in $G'$. Hence the subgraph $H$ induced by $S'$ in $G'$ is connected.

**(ii)** Since the subgraph $T$ induced by $S$ is a weighted tree cover of $G$, then for each edge $e = (v_k, v_l) \in E$ of $G$, $V(S)$ contains either $v_k$ or $v_l$ or both. By the construction of $G'$ from $G$, $G'$ has an $\alpha$-face for each edge $e \in E$ of $G$. Thus $V(S')$ contains $v_k$ or $v_l$ or both for each $\alpha$-face of graph $G'$. Since each edge of $G$ is covered by $V(S)$, each $\alpha$-face of graph $G'$ is covered by $V(S')$. We now need to show that the $\beta$-faces of $G'$ are also covered by $V(S')$. Since each $\beta$-face of $G'$ contains the original vertices of at least three $\alpha$-faces and $V(S')$ contains at least one original vertex from each $\alpha$-face, $V(S')$ contains at least two original vertices. Hence each $\beta$-face of $G'$ is covered by $V(S')$. Thus $V(S')$ covers all the faces of $G'$, that means, the subgraph $H$ induced by $S'$ in $G'$ cover all faces of $G'$.

**(iii)** The cost of $T$ is $\leq b$. Since $T = H$ and $b = b'$, the cost of $H$ is $\leq b'$ in $G'$. $\mathcal{Q.E.D.}$

**Proof of Lemma 2:** Since the construction of $G'$ from $G$ takes polynomial time, Lemma 3 implies that the face-spanning subgraph problem is *NP*-hard. $\mathcal{Q.E.D.}$

By Lemma 1 and 2, the following theorem holds.

**Theorem 1.** *The face-spanning subgraph problem is NP-complete.*

## 4 Minimum-Vertex Face-Spanning Subgraph Problem

In this section we consider a variation of the face-spanning subgraph problem, which we call minimum-vertex face-spanning subgraph problem. The formal definition of the problem is as follows:

**Definition 3.** (Minimum-Vertex Face-Spanning Subgraph Problem) *Let* $G = (V, E)$ *be a connected plane graph , where* $V$ *and* $E$ *are the set of vertices and edges, respectively, and let* $F$ *be the set of faces of graph* $G$*. Then is there any set* $S \subseteq E$ *such that the subgraph* $H$ *induced by* $S$ *is connected, cover all faces of* $G$ *and* $|V(H)| \leq k$*, for a given positive integer* $k \leq |V|$*?*

The minimum-vertex face-spanning subgraph problem often arises in applications like establishing base transceiver station in wireless networks, establishing power distribution centers in a city etc where the setup cost for each establishment is huge. In these cases the objective is to minimize the number of vertices instead of edge cost. To prove that the minimum-vertex face-spanning subgraph problem is *NP*-complete, we use the well known *NP*-complete problem "connected vertex cover" problem. The subgraph $H$ be a *connected vertex cover* of $G$ if $V(H)$ is a vertex cover of $G$ and the subgraph $H$ induced by $V(H)$ is connected. The formal definition of connected vertex cover problem is as follows [GJ77].

**Definition 4.** (Connected Vertex Cover Problem) *Given a plane graph* $G = (V, E)$ *and an integer* $k$*, does there exist a vertex cover* $V' \subseteq V$ *satisfying* $|V'| \leq k$ *and the subgraph induced by* $V'$ *is connected?*

In the rest of this section we prove that the minimum-vertex face-spanning subgraph problem is *NP*-complete. To prove that the minimum-vertex face-spanning subgraph problem is *NP*-complete, we show that (i) the minimum-vertex face-spanning subgraph problem is in *NP* and (ii) the minimum-vertex face-spanning subgraph problem is *NP*-hard.

As the first step of the proof, we have the following lemma.

**Lemma 4.** *The minimum-vertex face-spanning subgraph problem is in NP.*

*Proof.* The proof is similar to the proof of Lemma 1. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

We now prove the following lemma as the second part of the proof.

**Lemma 5.** *The minimum-vertex face-spanning subgraph problem is NP-hard.*

Let $G = (V, E)$ be a connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively. Let $F$ be the set of faces of graph $G$. We obtain a plane graph $G'$ from $G$ using the construction described in Section 3 as illustrated in Figure 3. Note that we do not consider weight of the edges, since the graph is not weighted in the current problem.

We now have the following lemma.

**Lemma 6.** $G'$ *has a minimum-vertex face-spanning subgraph* $H'$ *with* $|V(H')| \leq k'$ *if and only if* $G$ *has a connected vertex cover* $H$ *with* $|V(H)| \leq k$*, where* $k$ *and* $k'$ *are two positive integers.*

*Proof.* The proof is similar to the proof of Lemma 3. The detail of the proof is omitted in this extended abstract. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

**Proof of Lemma 5:** Since the construction of $G'$ from $G$ takes polynomial time, Lemma 6 implies that the minimum-vertex face-spanning subgraph problem is *NP*-hard. $\mathcal{Q.E.D.}$

By Lemma 4 and 5, the following theorem holds.

**Theorem 2.** *The minimum-vertex face-spanning subgraph problem is NP-complete.*

## 5 Approximation Algorithms

In this section we discuss some issues related to the approximation algorithms for finding the minimum face-spanning subgraph and the minimum-vertex face-spanning subgraph.

In practical applications of the face-spanning subgraph problem like the gas pipelines planning problem in Section 1, an input is often a plane graph $G$ such that each vertex of $G$ has degree three or more. We thus consider graphs of the minimum degree three in this section for designing approximation algorithms.

We first establish a lower bound on the number of vertices of a face-spanning subgraph of a plane graph. Let $H$ be a face-spanning subgraph of $G$ induced by edge set $S \subseteq E$. We call $H$ a *minimal face-spanning subgraph* of $G$ if there is no edge set $S' \subseteq S$ such that the subgraph induced by $S'$ is a face-spanning subgraph of $G$. Clearly a minimal face-spanning subgraph is a tree. Figure 5 illustrates an example of minimal face-spanning subgraph. The thick lines in Figure 5(a) is a minimal face-spanning subgraph. The thick lines in Figure 5(b) is not a minimal face-spanning subgraph since the subset of this thick lines can induce a face-spanning subgraph.

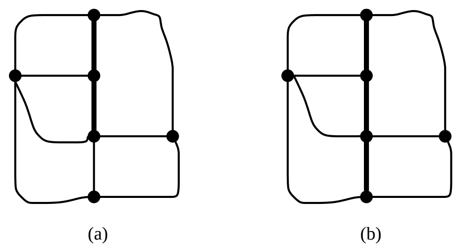


**Fig. 5.** Illustration of (a) a minimal face-spanning subgraph, and (b) a non-minimal face-spanning subgraph.

A plane graph may have many minimal face-spanning subgraphs. Note that a minimum face-spanning subgraph of $G$ defined in Section 2 is one of the minimal face-spanning subgraphs of $G$ whose total edge weight is minimum among all the minimal face-spanning subgraphs. Thus we can find a minimum face-spanning subgraph of a plane graph $G$ by finding all minimal face-spanning subgraphs and choosing one whose total edge weight is minimum among all the minimal face-spanning subgraphs of $G$.

A minimum-vertex face-spanning subgraph $H$ of $G$ defined in Section 4 may not be a minimal face-spanning subgraph of $G$, since the definition of a minimum-vertex face-spanning subgraph allows cycles in $H$. However, there exists a minimal face-spanning subgraph $H'$ of $G$ with the vertex set $V(H)$, and $H'$ can be obtained by removing an edge from each cycle in $H$ if $H$ has any cycle.

We now have the following lemma regarding the lower bound on the number of vertices of a minimal face-spanning subgraph whose proof is omitted in this extended abstract.

**Lemma 7.** *Let $G = (V, E)$ be a connected plane graph. Assume that each vertex of $G$ has degree three or more. Let $H$ be a minimal face-spanning subgraph induced by $S \subseteq E$ of $G$. Then $|V(S)| \geq (f-2)/(\Delta-2)$, where $f$ is the number of faces of $G$.*

We have a graph of 9 faces with $\Delta = 3$ as illustrated in Figure 6, for which the minimum number of vertices required for a face-spanning subgraph is 7. Thus the example in Figure 6 attains the lower bound, and hence the bound is tight.

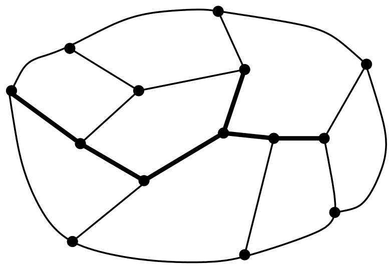

**Fig. 6.** A graph of 9 faces with $\Delta = 3$ for which the minimum number of vertices required for a face-spanning subgraph is 7.

We now give an algorithm for finding a minimal face-spanning subgraph based on spanning tree. Let $G = (V, E)$ be a connected plane graph, where $V$ and $E$ are the set of vertices and edges, respectively, and let $F$ be the set of faces of $G$. Let $v_0$ be an outer vertex of $G$. Let $G'$ be the graph obtained from $G$ by deleting all outer vertices of $G$ except $v_0$. Let $T$ be a spanning tree of $G'$. One can observe that $T$ is a face-spanning subgraph of $G$. We traverse the tree $T$ and delete each leaf vertex $v$ of $T$ if each of the faces of $G$ which contains $v$ is covered by any other vertex in $T$. Deletion of $v$ from $T$ may generate a new leaf vertex of $T$. We repeat the operation above for all the leaf vertices of $T$ including the newly generated leaf vertices. The resulting tree $T$ is our desired minimal face-spanning subgraph $H$. Using a data structure similar to that described in

Lemma 1 we can obtain a minimal face-spanning subgraph mentioned above in linear time. We call the algorithm described above ***Find-Minimal-Subgraph***.

Clearly the following lemma holds on the upper bound of the number of vertices of a minimal face-spanning subgraph produced by Algorithm *Find-Minimal-Subgraph*.

**Lemma 8.** *Let $G$ be a plane graph of $n$ vertices, and let $n_0$ be the number of outer vertices of $G$. Assume that each vertex of $G$ has degree three or more. Then Algorithm Find-Minimal-Subgraph produce a minimal face-spanning subgraph with at most $n - n_0 + 1$ vertices in linear time.*

The upper bound in Lemma 8 is also tight, since we have an infinite number of examples attaining the bound; one example is shown in Figure 7.

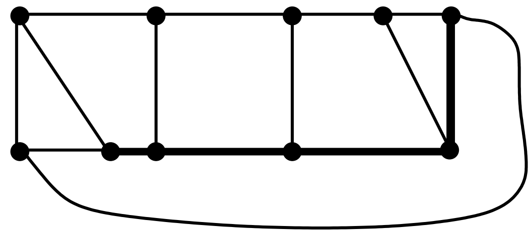

**Fig. 7.** A graph of 7 faces with $n = 10$ and $n_0 = 6$ for which the minimum number of vertices required for a face-spanning subgraph is 5.

We can take a minimal face-spanning subgraph of a connected plane graph $G$ produced by Algorithm *Find-Minimal-Subgraph* as an approximate solution of the minimum-vertex face-spanning subgraph problem, then we have the following theorem.

**Theorem 3.** *Let $G = (V, E)$ be a connected plane graph. Then the approximation ratio of Algorithm Find-Minimal-Subgraph for finding a minimum-vertex face-spanning subgraph is $2(\Delta - 2)$.*

*Proof.* Algorithm *Find-Minimal-Subgraph* constructs a minimum-vertex face-spanning subgraph of $G$ with at most $n - n_0 + 1$ vertices. By Lemma 7 a face-spanning subgraph of $G$ has at least $(f-2)/(\Delta-2)$ vertices. Hence approximation ratio is $(n-n_0+1)/\{(f-2)/(\Delta-2)\}$. From Euler's Formula for planar graphs, we have $f-2 = m-n$. Since degree of any vertex in $G$ is $\geq 3$, $2m \geq 3n$. This implies $(m-n) \geq n/2$ and hence $(f-2) \geq n/2$. Therefore the approximation ratio is $(n - n_0 + 1)/\{(f-2)/(\Delta-2)\} \leq (n - n_0 + 1)/\{(n/2)/(\Delta-2)\} = 2(n - n_0 + 1)(\Delta - 2)/n \leq 2(\Delta - 2)$. $\mathcal{Q.E.D.}$

A minimal face-spanning subgraph produced by Algorithm *Find-Minimal-Subgraph* can also be taken as an approximate solution of the minimum face-spanning subgraph problem. One can easily observe that approximation ratio of

Algorithm *Find-Minimal-Subgraph* for finding minimum face-spanning subgraph is $\{(n-n_0)e_{max}\}/\{\{(f-2)/(\Delta-2)-1\}e_{min}\} = \{(n-n_0)(\Delta-2)e_{max}\}/\{(f-\Delta)e_{min}\}$, where $e_{max}$ and $e_{min}$ denote the maximum and minimum weight of the edges of $G$.

## 6 Conclusion

In this paper we showed that the face-spanning subgraph problem and the minimum-vertex face-spanning subgraph problem are *NP*-complete. Thus it is unlikely to develop efficient algorithms for these problems. Since the problems arise from many practical applications, developing efficient approximation algorithms are essential. We have designed approximation algorithms for both the minimum face-spanning subgraph problem and the minimum-vertex face-spanning subgraph problem. We also analyzed the complexities and approximation ratios of the designed approximation algorithms. We have shown a lower bound for both the problems based on the number of vertices which is tight. We have also shown a tight upper bound of the number of vertices of a minimal face-spanning subgraph. However, to design approximation algorithms with better approximation ratio for the face-spanning subgraph problem and the minimum-vertex face-spanning subgraph problem is left as open problems.

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