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011

ON THE SPREAD OF CONTINUOUS-TIME LINEAR SYSTEMS*

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Abstract. Given the impulse response h of a linear time invariant system, this paper considers signals $y = h * u$ with inputs u subject to $|u(t)| \leqq 1$ and asks, for a given $\tau > 0$ and $y(t_0)$, what is the set of all the possible values (the "spread") of $y(t_0 + \tau)$. This set is characterized, its properties are studied, and it is computed for some functions h.

Key words. control, input, output, signal, spread of linear systems

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1. Basic definitions and results. Let $h(t)$ be a prescribed continuous function defined for $0 \leqq t < \infty$ and belonging to $L^1(0, \infty)$; we refer to it as *an impulse response.* Let $u(t)$ be any measurable function for $0 \leqq t < \infty$ satisfying $|u(t)| \leqq 1$; we refer to it as an *input*. The function $y(t)$, defined

$$y(t) = \int_0^t h(t-s)u(s)\, ds,$$

is called the *output* or the *signal.*

Given $\alpha \in \mathbb{R}$, $t_0 \geqq 0$, $\tau > 0$, we would like to estimate the range of the output $y(t)$ at time $t = t_0 + \tau$, given that $y(t_0) = \alpha$. More quantitatively, we wish to bound the numbers

$$\tilde{\sigma}^+(\alpha, \tau, t_0) = \sup_u \{y(t_0 + \tau); \text{given } y(t_0) = \alpha\},$$

$$\tilde{\sigma}^-(\alpha, \tau, t_0) = \inf_u \{y(t_0 + \tau); \text{given } y(t_0) = \alpha\}.$$

Introduce the class of control functions

$$K_{\tau,\alpha} = \left\{ u \in L^\infty(-\infty, \tau), -1 \leqq u(s) \leqq 1, \int_{-\infty}^0 h(-s)u(s)\, ds = \alpha \right\} \tag{1.1}$$

and the functional

$$J_\tau(u) = \int_{-\infty}^\tau h(\tau - s)u(s)\, ds, \tag{1.2}$$

and define

$$\sigma^+(\tau, \alpha) = \sup_{u \in K_{\tau,\alpha}} J_\tau(u), \tag{1.3}$$

$$\sigma^-(\tau, \alpha) = \inf_{u \in K_{\tau,\alpha}} J_\tau(u). \tag{1.4}$$

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DEFINITION 1.1. The function $\sigma(\tau, \alpha) = \sigma^+(\tau, \alpha) - \sigma^-(\tau, \alpha)$ is called the *spread of the linear system.*

The motivation comes from the following theorem.

THEOREM 1.1. *For any* $\alpha \in \mathbb{R}$, $\tau > 0$,

$$\sup_{t_0 \geqq 0} \tilde{\sigma}^+(\tau, \alpha, t_0) = \sigma^+(\tau, \alpha), \tag{1.5}$$

$$\inf_{t_0 \geqq 0} \tilde{\sigma}^-(\tau, \alpha, t_0) = \sigma^-(\tau, \alpha), \tag{1.6}$$

and, consequently,

$$\sup_{t_0 \geqq 0} \tilde{\sigma}^+(\tau, \alpha, t_0) - \inf_{t_0 \geqq 0} \tilde{\sigma}^-(\tau, \alpha, t_0) = \sigma(\tau, \alpha). \tag{1.7}$$

Proof. The condition $y(t_0) = \alpha$ means that

$$\int_0^{t_0} h(t_0 - s) u(s)\, ds = \alpha. \tag{1.8}$$

Writing

$$y(t_0 + \tau) = \int_0^{t_0 + \tau} h(t_0 + \tau - s) u(s)\, ds$$

and substituting $t_0 - s = -s'$, $u(t_0 + s') = v(s')$, we get

$$y(t_0 + \tau) = \int_{-t_0}^{\tau} h(\tau - s') v(s')\, ds'.$$

The same substitution applied to (1.8) gives

$$\int_{-t_0}^{0} h(-s') v(s')\, ds' = \alpha.$$

Hence

$$\tilde{\sigma}^+(\tau, \alpha, t_0) = \sup \left\{ \int_{-t_0}^{\tau} h(\tau - s') v(s')\, ds'; \quad v \text{ satisfies } |v(s')| \leqq 1, \right.$$
$$\left. \int_{-t_0}^{0} h(-s') v(s')\, ds' = \alpha \right\}.$$

Extending $v(s')$ to $s' < -t_0$ by zero, we see that $\sigma^+(\tau, \alpha, t_0)$ is $\sup J_\tau(v)$ when v is restricted to a subset say K_{τ,α,t_0}, of $K_{\tau,\alpha}$; hence

$$\tilde{\sigma}^+(\tau, \alpha, t_0) \leqq \sigma^+(\tau, \alpha).$$

As $t_0 \to \infty$ the subsets K_{τ,α,t_0} increase and every $u \in K_{\tau,\alpha}$ restricted to a bounded interval is a function in $\bigcup_{t_0 > 0} K_{\tau,\alpha,t_0}$ restricted to the same interval; this implies the equality in (1.5). The proof of (1.6) is similar.

THEOREM 1.2. *For any* $\alpha \in \mathbb{R}$, $\tau > 0$ *there exist admissible functions* $u^+_{\tau,\alpha}$, $u^-_{\tau,\alpha}$ *in* $K_{\tau,\alpha}$ *such that*

$$J_\tau(u^+_{\tau,\alpha}) = \sup_{u \in K_{\tau,\alpha}} J_\tau(u) = \sigma^+(\tau, \alpha), \tag{1.9}$$

$$J_\tau(u^-_{\tau,\alpha}) = \inf_{u \in K_{\tau,\alpha}} J_\tau(u) = \sigma^-(\tau, \alpha). \tag{1.10}$$

Indeed, taking a maximizing sequence u_j, we can extract a subsequence that is weakly convergent in L^1_{loc} to a function u_0. It is easy to check that u_0 is a maximizer for J_τ, i.e., u_0 is the asserted $u^+_{\tau,\alpha}$. The proof of (1.10) is similar.

In this paper we study the structure of $u^\pm_{\tau,\alpha}$ and this enables us to compute the spread of some linear systems of interest. In § 2 we solve a general maximization problem, which is then used in § 3 to analyze the structure of $u^\pm_{\tau,\alpha}$. In § 4 we establish various properties of $\sigma(\tau, \alpha)$, and in § 5 we compute $\sigma(\tau, \alpha)$ for some examples. Finally, in § 6 we show that all the results can be extended to the case where $y(t_0)$, $y(t_0+\tau_1), \cdots, y(t_0+\tau_{N-1})$ are prescribed and the range of $y(t_0+\tau_N)$ is sought; here $0<\tau_1<\tau_2<\cdots<\tau_N$.

Motivation for studying the function $\sigma^\pm(\tau, \alpha)$ comes from the following problem, posed in [1] and [2]. For any $d>0$, $T>0$ and impulse response $h(\cdot)$, denote by $N_{\max}(T, d)$ the maximum number of inputs $u_j(s)$ such that the corresponding outputs $y_j(t)$ satisfy

$$\max_{0<t\leqq T} |y_i(t)-y_j(t)| \geqq d \quad \forall i \neq j.$$

Since the mapping $u \to y$, in $L^\infty(0, T)$, maps the set of inputs u into a compact subset, the number $N_{\max}(T, d)$ is finite. We define

$$MCT(d) = \lim_{T\to\infty} \frac{\log N_{\max}(T, d)}{T} \text{ bits/sec,} \tag{1.11}$$

and would like to obtain bounds on the $MCT(d)$ for any $h(\cdot)$. Set

$$\tau^* = \inf\{\tau \mid \sigma(\tau, 0) = d\}.$$

Work in progress [3] indicates that

$$MCT(d) \leqq \frac{1}{\tau^*} \tag{1.12}$$

for any $h(\cdot)$ that satisfies

$$\int_{\{h(\tau-s)/h(-s)\geqq 1\}} |h(-s)|\, ds \leqq \int_{\{h(\tau-s)/h(-s)\leqq 1\}} |h(-s)|\, ds$$

for all $\tau \leqq \tau^*$; the arguments used depend on results derived in this paper. Results obtained here (in § 6) for the N constraint problem, in which N output values are specified, can be used to tighten the upper bound given by (1.12) (see [3]).

The problem of computing spread for a discrete-time linear system with impulse response h_i, $i = 0, 1, 2, \cdots$, is considered in [2]. This computation is equivalent to solving a linear program with bounded variables and one equality constraint. Here we show how the spread can be computed for a continuous-time linear system. Two examples of special interest are presented in which the spread can be computed by finding a solution to a transcendental equation.

2. A general optimization problem. Let $f(s)$, $g(s)$ be continuous functions in $\infty < s \leqq 0$ that belong to $L^1(-\infty, 0)$, and assume that

$$f \neq 0 \quad \text{a.e.,} \tag{2.1}$$

$$\text{meas}\left\{\frac{g}{f} = \mu\right\} = 0 \quad \text{for any } \mu \in \mathbb{R}. \tag{2.2}$$

Let

$$K=\left\{u(s) \text{ measurable for } -\infty<s<0, |u(s)|\leqq 1, \int_{-\infty}^{0} f(s)u(s)\,ds=\alpha\right\}$$

for some fixed $\alpha\in\mathbb{R}$, and

$$J(u)=\int_{-\infty}^{0} g(s)u(s)\,ds.$$

As in the proof of Theorem 1.2, we can show that there exists a function $u_0\in K$ such that

$$J(u_0)=\max_{u\in K} J(u). \tag{2.3}$$

THEOREM 2.1. *For any solution $u_0\in K$ of (2.3) there exists a number $\lambda\in\mathbb{R}$ such that almost everywhere*

$$u_0(s)=\begin{cases}\operatorname{sgn} f(s) & \text{if } g(s)/f(s)>\lambda,\\ -\operatorname{sgn} f(s) & \text{if } g(s)/f(s)<\lambda.\end{cases} \tag{2.4}$$

Note that (2.4) is equivalent to

$$u_0(s)=\operatorname{sgn}[g(s)-\lambda f(s)].$$

Proof. We begin by proving that $u_0=1$ almost everywhere. If the assertion is not true then the set $G_0=\{|u_0|<1\}$ has positive measure. Denote by G the subset of G_0 consisting of all points t of G_0-density equal to 1, such that also $f(t)\neq 0$. Then meas $G=$ meas $G_0>0$.

Take t_1, t_2 in $G(t_1\neq t_2)$ and let G_i be a subset of G contained in the δ_0-neighborhood of t_i, such that $\sup_{G_i}|u|<1$, meas $G_i\neq 0$ and $2\delta_0<|t_1-t_2|$. By decreasing one of these sets we arrive at the situation where

$$G_1\cap G_2=\emptyset, \qquad \text{meas } G_1=\text{meas } G_2=\delta>0.$$

For any real numbers A_1, A_2, if ε is positive and small enough then the function

$$\tilde{u}=u_0+A_1\frac{\varepsilon}{\delta}\chi_{G_1}+A_2\frac{\varepsilon}{\delta}\chi_{G_2} \tag{2.5}$$

satisfies $|\tilde{u}|\leqq 1$. Furthermore, if

$$A_1\int_{G_1} f(s)\,ds+A_2\int_{G_2} f(s)\,ds=0, \tag{2.6}$$

then $\int_{-\infty}^{0} f(s)\tilde{u}(s)\,ds=\alpha$, so that $\tilde{u}\in K$. Note that (2.6) is equivalent to

$$A_1 f(t_1)+A_2 f(t_2)=\sigma_1(\delta_0) \tag{2.7}$$

for some $\sigma_1(\delta_0)$ such that $\sigma_1(\delta_0)\to 0$ if $\delta_0\to 0$.

From the maximality of u_0 it follows that (2.6), or (2.7), implies $J(\tilde{u})\leqq J(u_0)$, that is,

$$A_1\int_{G_1} g(s)\,ds+A_2\int_{G_2} g(s)\,ds\leqq 0, \tag{2.8}$$

i.e.,

$$A_1 g(t_1)+A_2 g(t_2)\leqq\sigma_2(\delta_0) \tag{2.9}$$

for some $\sigma_2(\delta_0)$ such that $\sigma_2(\delta_0)\to 0$ if $\delta_0\to 0$.

If we choose

$$A_1 = -A_2 \frac{f(t_2)}{f(t_1)} + \frac{\sigma_1(\delta_0)}{f(t_1)} \tag{2.10}$$

so that (2.7) is satisfied, (2.9) must then hold and, upon letting $\delta_0 \to 0$, we get

$$A_2 \left[-\frac{g(t_1) f(t_2)}{f(t_1)} + g(t_2) \right] \leqq 0. \tag{2.11}$$

Since A_2 is arbitrary, it follows that the expression in brackets must vanish. Thus

$$\frac{g(t_1)}{f(t_1)} = \frac{g(t_2)}{f(t_2)}$$

for all t_1, t_2 in G. Since G has a positive measure, this is a contradiction to (2.2).

Denote by D the set of all points t such that $f(t) \neq 0$ and t is a Lebesgue point of u_0. Thus almost all t in $(-\infty, 0)$ belong to D. Take any t_1, t_2 in D with

$$\frac{g(t_1)}{f(t_1)} > \frac{g(t_2)}{f(t_2)}. \tag{2.12}$$

We will prove that almost everywhere

$$u_0(t_2) = \operatorname{sgn} f(t_2) \text{ implies } u_0(t_1) = \operatorname{sgn} f(t_1), \tag{2.13}$$

$$u_0(t_1) = -\operatorname{sgn} f(t_1) \text{ implies } u_0(t_2) = -\operatorname{sgn} f(t_2). \tag{2.14}$$

These two statements clearly imply assertion (2.4).

To prove (2.13) suppose the assertion is not true. Then the set $\tilde{G}$ of the pair (t_1, t_2) for which (2.13) is not true has positive measure. Choose t_1, t_2 at which $\tilde{G}$ has density 1. Since t_1 and t_2 are Lebesgue points of the function $u_0(t)$ and $|u_0| = 1$ almost everywhere, for any $\delta_0 > 0$ we can find sets G_1, G_2 such that meas $G_i \neq 0$, G_i is contained in the δ_0-neighborhood of t_i, and

$$u_0(t) = \operatorname{sgn} f(t) \quad \text{for all } t \in G_2,$$
$$u_0(t) = -\operatorname{sgn} f(t) \quad \text{for all } t \in G_1.$$

By choosing $2\delta_0 < |t_1 - t_2|$ and by suitably decreasing one of the sets G_i, we get $G_1 \cap G_2 = \phi$, meas $G_1 =$ meas G_2. We again form the function (2.5). If

$$A_2 \operatorname{sgn} f(t_2) < 0, \qquad A_1 \operatorname{sgn} f(t_1) > 0, \tag{2.15}$$

then $|\tilde{u}| \leqq 1$ if ε is sufficiently small.

If we can further choose A_1, A_2 such that (2.6) (or (2.7)) holds, then (2.8) (or (2.9)) must be satisfied. Condition (2.7) is satisfied by the choice (2.10) of A_1, and if $A_2 \operatorname{sgn} f(t_2) < 0$, then clearly also $A_1 \operatorname{sgn} f(t_1) > 0$ provided δ_0 is sufficiently small. We conclude, after letting $\delta_0 \to 0$, that (2.11) must hold provided $A_2 \operatorname{sgn} f(t_2) < 0$. Dividing (2.11) by $A_2 f(t_2)$, we arrive at the inequality

$$-\frac{g(t_1)}{f(t_1)} + \frac{g(t_2)}{f(t_2)} \geqq 0,$$

which is a contradiction to (2.12). This completes the proof of (2.13); the proof of (2.14) is similar.

From Theorem 2.1 we immediately get Corollary 2.2.

COROLLARY 2.2. *The constant* λ *in Theorem* 2.1 *is uniquely determined by*

$$\int_{\{g(s)/f(s) > \lambda\}} |f(s)|\, ds - \int_{\{g(s)/f(s) < \lambda\}} |f(s)|\, ds = \alpha; \tag{2.16}$$

consequently the maximizer u_0 is also uniquely determined. As α decreases from $\int_{-\infty}^{0} |f(s)|\,ds$ to $-\int_{-\infty}^{0} |f(s)|\,ds$, $\lambda = \lambda(\alpha)$ increases monotonically from

$$\inf_{s<0} \{g(s)/f(s)\} \quad \text{to} \quad \sup_{s<0} \{g(s)/f(s)\}.$$

3. The structure of $u^{\pm}_{\tau,\alpha}$. *Choose* $h(t)$ as in § 1, i.e.,

$$h \in L^1(0,\infty) \cap C^0[0,\infty) \tag{3.1}$$

and assume further that

$$h(t) \neq 0 \quad \text{a.e.,} \tag{3.2}$$

$$\text{meas}\left\{0<t<\infty; \frac{h(t+\tau)}{h(t)} = \lambda\right\} = 0 \quad \forall \tau > 0, \quad \lambda \in \mathbb{R}. \tag{3.3}$$

Taking $f(t) = h(-t)$, $g(t) = h(\tau - t)$ in Theorem 2.1 and Corollary 2.2, we get Theorem 3.1.

THEOREM 3.1. *There exists a unique solution $u^+_{\tau,\alpha}$ of* (1.9) *given by*

$$u^+_{\tau,\alpha}(s) = \begin{cases} \text{sgn}\, h(-s) & \text{if } \dfrac{h(\tau-s)}{h(-s)} > \lambda^+, \\ -\text{sgn}\, h(-s) & \text{if } \dfrac{h(\tau-s)}{h(-s)} < \lambda^+ \end{cases} \tag{3.4}$$

where λ^+ is determined by

$$\int_{\{h(\tau-s)/h(-s)>\lambda^+\}} |h(-s)|\,ds - \int_{\{h(t-s)/h(-s)<\lambda^+\}} |h(-s)|\,ds = \alpha. \tag{3.5}$$

Clearly also $u^+_{\tau,\alpha}(s) = \text{sgn}\, h(\tau - s)$ if $0 < s < \tau$.

We now consider a special case.

THEOREM 3.2. *If $h \in L^1(0,\infty)$, $h > 0$, $d^2(\log h)/dt^2 > 0$, then there is a unique solution of* (1.9) *given by*

$$u^+_{\tau,\alpha}(s) = \begin{cases} 1 & \text{if } -\infty < s < \mu, \\ -1 & \text{if } \mu < s < 0 \end{cases} \tag{3.6}$$

and $u^+_{\tau,\alpha}(s) = 1$ if $0 < s < \tau$, where μ is determined by

$$\int_{-\infty}^{\mu} h(-s)\,ds - \int_{\mu}^{0} h(-s)\,ds = \alpha. \tag{3.7}$$

Proof. By assumption,

$$\frac{h'(s)}{h(s)} \text{ is strictly increasing;}$$

hence

$$\frac{h'(\tau+s)}{h(\tau+s)} > \frac{h'(s)}{h(s)}.$$

This means that

$$\frac{d}{ds}\frac{h(\tau+s)}{h(s)} > 0,$$

and thus

$$\frac{h(\tau-s)}{h(-s)} \text{ is strictly decreasing in } s.$$

Now apply Theorem 3.1 to complete the proof.

Remark 3.1. If $\log h$ is convex (but not satisfying $d^2(\log h)/dt^2>0$), then we can approximate it by a smooth function h_n with $d^2(\log h_n)/dt^2>0$. Applying Theorem 3.2 to the corresponding maximizers $u^{h_n}_{\tau,\alpha}$, we deduce that there is a maximizer $u_{\tau,\alpha}$ (for h) having the form (3.6), (3.7). There may be other maximizers; for instance, if $h(t)=e^{-t}$ then every $u\in K_{\tau,\alpha}$ is a maximizer. (Note that (3.3) does not hold for $h(t)=e^{-t}$.)

THEOREM 3.3. *If* $h\in L^1(0,\infty)$, $h>0$, *and* $d^2(\log h)/dt^2<0$, *then there is a unique solution of* (1.9) *given by*

$$u^+_{\tau,\alpha}(s)=\begin{cases}-1 & \text{if } -\infty<s<\tilde{\mu},\\ 1 & \text{if } \tilde{\mu}<s<0\end{cases} \tag{3.8}$$

and $u^+_{\tau,\alpha}(s)=1$ *if* $0<s<\tau$, *where* $\tilde{\mu}$ *is determined by*

$$-\int_{-\infty}^{\tilde{\mu}} h(-s)\,ds+\int_{\tilde{\mu}}^{0} h(-s)\,ds=\alpha. \tag{3.9}$$

Note that $d\mu/d\alpha>0$, $d\tilde{\mu}/d\alpha<0$, where $\mu=\mu(\alpha)$ and $\tilde{\mu}=\tilde{\mu}(\alpha)$ are defined by (3.7) and (3.9), respectively.

4. Properties of the spread. Theorem 3.1 implies that

$$\begin{aligned}\sigma^+(\tau,\alpha)&=\int_{\{h(\tau-s)/h(-s)>\lambda^+\}}[\operatorname{sgn} h(-s)]h(\tau-s)\,ds\\ &\quad-\int_{\{h(\tau-s)/h(-s)<\lambda^+\}}[\operatorname{sgn} h(-s)]h(\tau-s)\,ds+\int_0^\tau|h(\tau-s)|\,ds\\ &=\int_{\{h(\tau-s)/h(-s)>\lambda^+\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds\\ &\quad-\int_{\{h(\tau-s)/h(-s)<\lambda^+\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds+\int_0^\tau|h(\tau-s)|\,ds\end{aligned} \tag{4.1}$$

where λ^+ is determined by (3.5). Similarly, we can show that

$$\begin{aligned}\sigma^-(\tau,\alpha)&=-\int_{\{h(\tau-s)/h(-s)>\lambda^-\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds\\ &\quad+\int_{\{h(\tau-s)/h(-s)<\lambda^-\}}|h(-s)|\,ds-\int_0^\tau|h(\tau-s)|\,ds\end{aligned} \tag{4.2}$$

where λ^- is determined by

$$-\int_{\{h(\tau-s)/h(-s)>\lambda^-\}}|h(-s)|\,ds+\int_{\{h(\tau-s)/h(-s)<\lambda^-\}}|h(-s)|\,ds=\alpha. \tag{4.3}$$

As α decreases from $\int_0^\infty|h(s)|\,ds$ to $-\int_0^\infty|h(s)|\,ds$, $\lambda^-(\alpha)$ decreases monotonically from $\sup_{s<0}\{h(\tau-s)/h(-s)\}$ to $\inf_{s<0}\{h(t-s)/h(-s)\}$. Also, $\lambda^-(0)=\lambda^+(0)$.

Combining (4.1) and (4.2) gives the spread

$$\frac{1}{2}\sigma(\tau,\alpha)=\frac{1}{2}[\sigma^+(\tau,\alpha)-\sigma^-(\tau,\alpha)]$$

$$(4.4)\qquad =\int_{\{h(\tau-s)/h(-s)>\lambda_M\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds$$
$$-\int_{\{h(\tau-s)/h(-s)<\lambda_m\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds+\int_0^\tau|h(\tau-s)|\,ds$$

where $\lambda_M=\max(\lambda^-,\lambda^+)$ and $\lambda_m=\min(\lambda^-,\lambda^+)$.

THEOREM 4.1. *There holds*

$$(4.5)\qquad \frac{\partial\sigma^\pm(\tau,\alpha)}{\partial\alpha}=\lambda^\pm.$$

Proof. Since $\lambda^+(\alpha)$ is a monotonically decreasing function of α, we can write

$$(4.6)\qquad \int_{\{h(\tau-s)/h(-s)>\lambda^++\Delta\lambda\}}|h(-s)|\,ds-\int_{\{h(\tau-s)/h(-s)<\lambda^++\Delta\lambda\}}|h(-s)|\,ds=\alpha-\Delta\alpha$$

where $\Delta\lambda$, $\Delta\alpha$ are positive. Subtracting (4.6) from (4.1) gives

$$(4.7)\qquad 2\int_{\{\lambda^+<h(\tau-s)/h(-s)<\lambda^++\Delta\lambda\}}|h(-s)|\,ds=\Delta\alpha.$$

From (4.1), (4.6), and (4.7),

$$\sigma^+(\tau,\alpha)-\sigma^+(\tau,\alpha-\Delta\alpha)=2\int_{\{\lambda^+<h(\tau-s)/h(-s)<\lambda^++\Delta\lambda\}}\frac{h(\tau-s)}{h(-s)}|h(-s)|\,ds$$
$$=[\lambda^++\varepsilon(\Delta\lambda)]\Delta\alpha$$

where $\varepsilon(\Delta\lambda)\to0$ as $\Delta\lambda\to0$. Letting $\Delta\alpha\to0$ gives $\partial\sigma^+/\partial\alpha=\lambda^+$. A similar argument shows that $\partial\sigma^-/\partial\alpha=\lambda^-$.

THEOREM 4.2. (i) *$\sigma^+(\tau,\alpha)$ is concave in α, $\sigma^-(\tau,\alpha)$ is convex in α, and thus $\sigma(\tau,\alpha)$ is concave in α*:

(ii) *$\sigma^\pm(\tau,\alpha)=-\sigma^\pm(\tau,-\alpha)$ and therefore $\sigma(\tau,\alpha)=\sigma(\tau,-\alpha)$*,

(iii) *$\partial\sigma(\tau,\alpha)/\partial\alpha\leqq0$ if $\alpha>0$.*

Proof. Assertion (i) follows immediately from Theorem 4.1 and the fact that $\partial\lambda^+/\partial\alpha$ $(\partial\lambda^-/\partial\alpha)$ is negative (positive) for all α. Assertion (ii) is obvious from the definition of $\sigma^\pm$. Finally, since $\sigma(\tau,\alpha)$ is concave in α (by (i)) and $\partial\sigma(\tau,\alpha)/\partial\alpha=0$ at $\alpha=0$ (by (ii)), (iii) follows.

We now specialize to the case where either $\log h$ is convex, so that

$$(4.8)\qquad \sigma^+(\tau,\alpha)=\int_{-\infty}^{\mu}h(\tau-s)\,ds-\int_\mu^0h(\tau-s)\,ds+\int_0^\tau h(s')\,ds'$$

where μ is determined by (3.7), or $\log h$ is concave so that

$$(4.9)\qquad \sigma^+(\tau,\alpha)=-\int_{-\infty}^{\tilde\mu}h(\tau-s)\,ds+\int_{\tilde\mu}^0h(\tau-s)\,ds+\int_0^\tau h(s')\,ds'$$

where $\tilde\mu$ is determined by (3.9).

THEOREM 4.3. *If $h'<0$ and $\log h$ is convex or concave, then*

$$(4.10)\qquad \frac{\partial\sigma^\pm(\tau,\alpha)}{\partial\tau}>0.$$

Proof. If $\log h$ is convex, then from (4.8) we get

$$\frac{\partial\sigma^+(\tau,\alpha)}{\partial\tau}=-\int_{-\infty}^{\mu}\frac{d}{ds}h(\tau-s)\,ds+\int_{\mu}^{0}\frac{d}{ds}h(\tau-s)\,ds+h(\tau)$$
$$=2h(\tau)-2h(\tau-\mu)>0.$$

Similarly, if $\log h$ is concave then

$$\frac{\partial\sigma^+(\tau,\alpha)}{\partial\tau}=\int_{-\infty}^{\tilde{\mu}}\frac{d}{ds}h(\tau-s)\,ds-\int_{\tilde{\mu}}^{0}\frac{d}{ds}h(\tau-s)\,ds+h(\tau)$$
$$=2h(\tau-\tilde{\mu})>0.$$

Finally, the second inequality in (4.10) follows from the first inequality and Theorem 4.2(ii).

5. Examples. If $h(t)=\exp\{-k(t)\}$, where $k(t)\to\infty$, k convex (k concave), then $\log h$ is concave (convex). For $h(t)=(t+a)^b$ where $a>0$, $b>0$, $\log h$ is convex.

We now consider two functions $h(t)$ of special interest.

THEOREM 5.1. *Let*

$$h(t)=\sum_{i=1}^{N}a_i\,e^{-\beta_i t}\qquad(a_i>0,\beta_i>0). \tag{5.1}$$

Then $d^2\log h/dt^2>0$.

Proof. As in the proof of Theorem 3.2, the assertion is equivalent to showing that

$$\frac{d}{ds}\frac{h(\tau-s)}{h(-s)}=\frac{h(-s)\Sigma a_i\beta_i\,e^{-\beta_i(\tau-s)}-h(\tau-s)\Sigma a_i\beta_i\,e^{\beta_i s}}{h^2(-s)}$$

is negative for any $\tau>0$. But the numerator is equal to

$$\sum\sum a_i\beta_i a_j(e^{-\beta_i(\tau-s)+\beta_j s}-e^{-\beta_j(\tau-s)+\beta_i s})$$
$$=\sum\sum a_i a_j\beta_i\,e^{s(\beta_i+\beta_j)}(e^{-\beta_i\tau}-e^{-\beta_j\tau})$$
$$=\tfrac{1}{2}\sum\sum a_i a_j\,e^{s(\beta_i+\beta_j)}[\beta_i(e^{-\beta_i\tau}-e^{-\beta_j\tau})+\beta_j(e^{-\beta_j\tau}-e^{-\beta_i\tau})]$$
$$=\tfrac{1}{2}\sum\sum a_i a_j\,e^{s(\beta_i+\beta_j)}(\beta_i-\beta_j)(e^{-\beta_i\tau}-e^{-\beta_j\tau})$$

and each term in the last sum is negative if $\beta_i\neq\beta_j$.

For the function (5.1), the μ determined by (3.7) is given by

$$\sum_{i=1}^{N}\frac{a_i}{\beta_i}(2\,e^{\beta_i\mu}-1)=\alpha.$$

The next example is

$$h(t)=e^{-\beta t}\cos\omega t\qquad(\beta>0,\omega>0). \tag{5.2}$$

Since

$$\frac{h(\tau-s)}{h(-s)}=e^{-\beta\tau}(\cos\omega\tau+\sin\omega\tau\tan\omega s),$$

we can check that the optimal solution $u^+_{\tau,\alpha}$, which for simplicity we will denote by u_0, satisfies

$$u_0(s)=\begin{cases}\operatorname{sgn} h(-s) & \text{if } \gamma-n\pi<\omega s<-\dfrac{(2n+1)\pi}{2},\\ -\operatorname{sgn} h(-s) & \text{if } -\dfrac{(2n+3)\pi}{2}<\omega s<\gamma-n\pi\end{cases}$$

if $n=0,1,2,\cdots$, and

$$u_0(s)=\begin{cases}-\operatorname{sgn} h(-s) & \text{if } -\pi/2<\omega s<\min(\gamma+\pi,0),\\ \operatorname{sgn} h(-s) & \text{if } \min(\gamma+\pi,0)<\omega s\leqq 0\end{cases}$$

where $\gamma\in[-3\pi/2,-\pi/2]$ is to be selected such that

$$\int_{-\infty}^{0} h(-s)u_0(s)\,ds=\alpha. \tag{5.3}$$

Recalling (5.2) we can check that

$$u_0(s)=\begin{cases}-1 & \text{if } \gamma-2n\pi<\omega s<\gamma-(2n-1)\pi,\\ 1 & \text{if } \gamma-(2n-1)\pi<\omega s<\gamma-2(n-1)\pi\end{cases}$$

for $n=1,2,\cdots$, and

$$u_0(s)=\begin{cases}-1 & \text{if } \gamma<\omega s<\min(\gamma+\pi,0),\\ 1 & \text{if } \min(\gamma+\pi,0)<\omega s<0.\end{cases}$$

Setting $\gamma'=\min(\gamma+\pi,0)$ and using the formula

$$\int_a^b h(-s)\,ds=-\operatorname{Re}\left\{\frac{1}{\beta+i\omega}\left[e^{-(\beta+i\omega)b}-e^{-(\beta+i\omega)a}\right]\right\},$$

we can compute

$$\int_{-\infty}^{0} h(-s)u_0(s)\,ds=\sum_{n=1}^{\infty}\left[-\int_{(\gamma-2n\pi)/\omega}^{(\gamma-(2n-1)\pi)/\omega} h(-s)\,ds+\int_{(\gamma-(2n-1)\pi)/\omega}^{(\gamma-2(n-1)\pi)/\omega} h(-s)\,ds\right]$$
$$-\int_{\gamma/\omega}^{\gamma'/\omega} h(-s)\,ds+\int_{\gamma'/\omega}^{0} h(-s)\,ds.$$

After somewhat lengthy calculations we get the expression

$$\operatorname{Re}\left\{\frac{\beta-i\omega}{\beta^2+\omega^2}e^{(\beta+i\omega)\gamma/\omega}\,\frac{1+e^{-\beta\pi/\omega}}{1-e^{-\beta\pi/\omega}}+\frac{\beta-i\omega}{\beta^2+\omega^2}\left[1-2\,e^{(\beta+i\omega)\gamma'/\omega}+e^{(\beta+i\omega)\gamma/\omega}\right]\right\}, \tag{5.4}$$

or

$$\frac{\beta}{\beta^2+\omega^2}+\frac{2\,e^{\beta\gamma/\omega}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}(\beta\cos\gamma+\omega\sin\gamma)-\frac{2\,e^{\beta\gamma'/\omega}}{\beta^2+\omega^2}(\beta\cos\gamma'+\omega\sin\gamma').$$

Hence (5.3) determines γ by the following formulas:

$$\frac{2\,e^{\beta\gamma/\omega}}{1-e^{-\beta\pi/\omega}}(\beta\cos\gamma+\omega\sin\gamma)=\alpha(\omega^2+\beta^2)+\beta \quad \text{if } -\pi<\gamma<0,$$

$$\left[\frac{2\,e^{\beta\gamma/\omega}}{1-e^{-\beta\pi/\omega}}+2\,e^{\beta(\gamma+\pi)/\omega}\right](\beta\cos\gamma+\omega\sin\gamma) \tag{5.5}$$
$$=\alpha(\omega^2+\beta^2)-\beta \quad \text{if } -3\pi/2<\gamma<-\pi.$$

Since

$$\int_{-\infty}^{0} h(\tau-s)u_0(s)\,ds=\operatorname{Re}\left\{\int_{-\infty}^{0} e^{-(\beta+i\omega)(\tau-s)}u_0(s)\,ds\right\}$$
$$=\operatorname{Re}\left\{e^{-(\beta+i\omega)\tau}\int_{-\infty}^{0} h_0(-s)u_0(s)\,ds\right\}$$

and the last integral is equal to the expression in braces in (5.4), we find that

$$\sigma^+(\tau,\alpha)=\frac{2\,e^{\beta((\gamma/\omega)-\tau)}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}[\beta\cos(\gamma-\omega\tau)+\omega\sin(\gamma-\omega\tau)]$$
$$-\frac{e^{-\beta\tau}}{\beta^2+\omega^2}(\beta\cos\omega\tau-\omega\sin\omega\tau)+\int_0^{\tau}|h(s)|\,ds \quad \text{if } -\pi<\gamma<0,$$

$$\sigma^+(\tau,\alpha)=\frac{2\,e^{\beta((\gamma+\pi)/\omega-\tau)}}{(\beta^2+\omega^2)(1-e^{-\beta\pi/\omega})}[\beta\cos(\gamma-\omega\tau)+\omega\sin(\gamma-\omega\tau)]$$
$$+\frac{e^{-\beta\tau}}{\beta^2+\omega^2}(\beta\cos\omega\tau-\omega\sin\omega\tau)+\int_0^{\tau}|h(s)|\,ds \quad \text{if } -\frac{3\pi}{2}<\gamma<-\pi.$$

6. Several constraints. The results of the previous sections can be extended to the case of several constraints. In fact it all hinges on generalizing Theorem 2.1 to the problem

$$\max_{u\in K_\alpha}\int_{-\infty}^{0} g(s)u(s)\,ds \tag{6.1}$$

where K_α is the set of all measurable functions $u(s)$ satisfying

$$-1\leqq u(s)\leqq 1 \quad \text{for } -\infty<s\leqq 0, \tag{6.2}$$

$$\int_{-\infty}^{0} f_i(s)u(s)\,ds=\alpha_i \qquad (i=1,2,\cdots,N). \tag{6.3}$$

Here g and f_i are given functions in $L^1(-\infty,0)\cap C^0(-\infty,0]$ and α_i are given real numbers.

THEOREM 6.1. *Assume that $f_1\neq 0$ almost everywhere and that, for any real numbers $\mu_1,\cdots,\mu_N$,*

$$\text{measure}\left\{g=\sum_{i=1}^{N}\mu_i f_i\right\}=0.$$

Then there exist sequences u_m, $\lambda_{i,m}$, $\alpha_{i,m}$ with $u_m\to u_0$ weakly in L^1_{loc}, $\alpha_{1,m}=\alpha_1$, $\alpha_{i,m}\to\alpha_i$ for $2\leqq i\leqq N$, where u_0 is a maximizer of (6.1), *and*

$$u_m(s)=\operatorname{sgn}\left[g(s)-\sum_{i=1}^{N}\lambda_{i,m}f_i(s)\right], \tag{6.4}$$

$$\int_{-\infty}^{0} f_i(s)u_m(s)\,ds = \alpha_{i,m} \qquad (i=1,2,\cdots,N). \tag{6.5}$$

Thus to evaluate (6.1) we need to analyze the u_m from (6.4), (6.5) and then compute $\int_{-\infty}^{0} gu_m$, noting that

$$\int_{-\infty}^{0} g(s)u_m(s)\,ds \to \int_{-\infty}^{0} g(s)u_0(s)\,ds = \max_{u\in K_\alpha} \int_{-\infty}^{0} g(s)u(s)\,ds.$$

Proof. For any small $\eta>0$ introduce the "penalized" functional

$$J_\eta(u) = \int_{-\infty}^{0} g(s)u(s)\,ds - \frac{1}{\eta}\sum_{i=2}^{N}\left[\int_{-\infty}^{0} f_i(s)u(s)\,ds - \alpha_i\right]^2 \tag{6.6}$$

and consider the problem

$$\text{maximize } J_\eta(u) \quad \text{for } u\in K \tag{6.7}$$

where K consists of all functions u satisfying

$$-1 \leqq u(s) \leqq 1, \qquad \int_{-\infty}^{0} f_1(s)\,ds = \alpha_1.$$

Proceeding as in the proof of Theorem 2.1, we deduce that if $|\tilde{u}|\leqq 1$, where $\tilde{u}$ is defined by (2.5) with $u_0=u_\eta$, then (2.6) implies

$$A_1 g(t_1) + A_2 g(t_2) - \frac{2}{\eta}\sum_{i=2}^{N}\left(\int_{-\infty}^{0} f_i u_\eta\,ds - \alpha_i\right)(A_1 f_i(t_1) + A_2 f_i(t_2)) \leqq \sigma_2(\delta_0)$$

where u_η is a solution of (6.7) and $\sigma_2(\delta_0)\to 0$ if $\delta_0\to 0$. Taking $\delta_0\to 0$, we get the inequality

$$A_1\left[g(t_1) - \sum_{i=2}^{N} \lambda_{i,\eta} f_i(t_1)\right] + A_2\left[g(t_2) - \sum_{i=2}^{N}\lambda_{i,\eta} f_i(t_2)\right] \leqq 0$$

for some scalars $\lambda_{i,\eta}$. We can now proceed as in § 2 to deduce that meas $\{|u_\eta|<1\}=0$; furthermore,

$$u_\eta(s) = \operatorname{sgn}\left[g(s) - \sum_{i=1}^{N}\lambda_{i,\eta} f_i(s)\right]. \tag{6.8}$$

We note that

$$J_\eta(u_\eta) \geqq J_\eta(\hat{u}) \quad \forall\, \hat{u}\in K;$$

from this inequality it follows that

$$\frac{1}{\eta}\sum_{i=2}^{N}\left[\int_{-\infty}^{0} f_i(s)u_\eta(s)\,ds - \alpha_i\right]^2 \leqq C, \quad C \text{ independent of } \eta.$$

Hence, as $\eta\to 0$,

$$\alpha_{i,\eta} \equiv \int_{-\infty}^{0} f_i(s)u_\eta(s)\,ds \to \alpha_i \qquad (2\leqq i\leqq N). \tag{6.9}$$

It is also easy to verify that for any convergent subsequence u_{η_m} (weakly in L^1_{loc}), the limit u_0 is a solution to problem (6.1). Indeed

$$J_\eta(u_\eta) \leqq \int_{-\infty}^{0} gu_\eta\,ds \leqq \max_{u\in K_{\alpha_\eta}} \int_{-\infty}^{0} gu\,ds = \int_{-\infty}^{0} g\hat{u}_\eta\,ds \tag{6.10}$$

where K_{α_η} is defined as K_α but with

$$\alpha_2 = \alpha_{2,\eta},\cdots,\alpha_N = \alpha_{N,\eta}.$$

Since $\alpha_{j,\eta} \to \alpha_j$, if we take η to vary in a subsequence of η_m such that $\hat{u}_\eta \to \hat{u}$ weakly in L^1_{loc}, then $\int_{-\infty}^0 g\hat{u}_\eta \to \int_{-\infty}^0 g\hat{u}$ and $\hat{u} \in K_\alpha$ (i.e., $\hat{u}$ satisfies (6.2), (6.3)). Denoting by u_1 any solution of (6.1)-(6.3), we then have

$$\int_{-\infty}^0 g\hat{u}\, ds \leqq \int_{-\infty}^0 gu_1\, ds;$$

also, by maximality of u_η (see (6.7)),

$$\int_{-\infty}^0 gu_1\, ds = J_\eta(u_1) \leqq J_\eta(u_\eta).$$

Using these relations in (6.10) and noting that

$$\int_{-\infty}^0 gu_\eta\, ds \to \int_{-\infty}^0 gu_0\, ds,$$

we conclude that

$$\int_{-\infty}^0 gu_0\, ds = \int_{-\infty}^0 gu_1\, ds = \max_{u \in K_\alpha} \int_{-\infty}^0 gu\, ds.$$

Thus u_0 is a solution to (6.1)-(6.3). Recalling (6.8), (6.9) completes the proof of Theorem 6.1.

Remark 6.1. The $\lambda_{i,m}$ satisfy

$$\int_{\{g > \Sigma \lambda_{j,m} f_j\}} f_i(s)\, ds - \int_{\{g < \Sigma \lambda_{j,m} f_j\}} f_i(s)\, ds = \alpha_i \qquad (1 \leqq i \leqq N).$$

From these equations we should be able to determine the $\lambda_{j,m}$, at least in some relatively simple examples, and show that $\lambda_{j,m} \to \lambda_j$ (λ_j finite) as $m \to \infty$; this would imply that

$$u_0 = \operatorname{sgn}\left[g(s) - \sum_{i=1}^N \lambda_i f_i(s) \right],$$

$$\int_{-\infty}^0 f_j(s) u_0(s)\, ds = \alpha_j \quad \text{for } 1 \leqq j \leqq N.$$

Remark 6.2. Theorem 2.1 can actually also be proved using the penalized functional

$$\int_{-\infty}^0 gu\, ds - \frac{1}{\eta}\left(\int_{-\infty}^0 fu\, ds - \alpha \right)^2 - \int_{-\infty}^0 \frac{(u-u_0)^2}{1+s^2}\, ds.$$

Remark 6.3. Consider the problem

$$\max_{u \in K} J_\tau(u) \tag{6.11}$$

where K is the set of all inputs u that satisfy

$$\int_{-\infty}^{t_j} h(t_j - s) u(s)\, ds = \alpha_j, \qquad j = 1, \cdots, N \tag{6.12}$$

and where $J_\tau(u)$ is defined by (1.2) and $0 = t_1 < t_2 < \cdots < t_{N-1} < t_N \equiv \tau$.

Set

$$\sigma^+(\tau; t_1, \alpha_1, \cdots, t_N, \alpha_N) = \max_{u \in K} J_\tau(u), \tag{6.13a}$$

$$\sigma^-(\tau; t_1, \alpha_1, \cdots, t_N, \alpha_N) = \min_{u \in K} J_\tau(u). \tag{6.13b}$$

Then there exists a solution to (6.11) if and only if

$$
(6.14)\qquad
\begin{aligned}
&|\alpha_1| \leqq \int_0^\infty |h(s)|\, ds,\\
&\sigma^-(t_j; t_1, \alpha_1, \cdots, t_{j-1}, \alpha_{j-1}) \leqq \alpha_j \leqq \sigma^+(t_j; t_1, \alpha_1, \cdots, t_{j-1}, \alpha_{j-1}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 < j \leqq N.
\end{aligned}
$$

If we assume $h(s) = 0$ for $s < 0$, the conclusion of Remark 6.1 becomes

$$
(6.15)\qquad u(s) = \begin{cases} \operatorname{sgn}\left[h(\tau - s) - \sum_{i=1}^{j} \lambda_i h(t_i - s) \right], & t_j < s < t_{j+1}, \\ \operatorname{sgn}\left[h(\tau - s) - \sum_{i=1}^{N} \lambda_i h(t_i - s) \right], & s < t_1 \end{cases}
$$

where $\lambda_1, \cdots, \lambda_N$ satisfy (6.12).

REFERENCES

[1] M. L. HONIG, S. BOYD, AND B. GOPINATH, *On optimum signal sets for digital communications with finite precision and amplitude constraints*, in Proc. IEEE Globecom. Conference, Tokyo, November 1987.

[2] M. L. HONIG, K. STEIGLITZ, AND B. GOPINATH, *Bounds on maximum throughput for digital communications with finite-precision and amplitude constraints*, in Proc. Internat. Conf. Acoustics, Speech, and Signal Processing, New York, NY, April 1988.

[3] M. L. HONIG, K. STEIGLITZ, S. BOYD, AND B. GOPINATH, *Bounds on maximum throughput for digital communications with finite-precision and amplitude constraints*, IEEE Trans. Inform. Theory, to appear.