Amortized Analysis

EECS 214

November 11–13, 2015
Take-aways

- What is *amortized time*?
- How does amortized time differ from *average time*?
- When is amortized time useful, and when might we want to avoid it?
- How can we figure out the amortized time of data structure operations?
- How does a dynamic array achieve its amortized time complexity?
Example: dynamic arrays

<table>
<thead>
<tr>
<th>Language</th>
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<tbody>
<tr>
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Iteratively growing a dynamic array

```cpp
std::vector<int> v;
for (int i = 0; i < N; ++i) v.push_back(i);

ArrayList<Integer> v = new ArrayList<>();
for (int i = 0; i < N; ++i) v.add(i);

v = list()
for i in range(0, 10): v.append(i)

v = Array.new
for i in 0 ... N do v.push(i) end
```
Time per operation

5:1
What’s it doing?

- A dynamic array is backed by a fixed-size array with excess capacity:
  \[
  \text{(define-struct dynarray [data size])}
  \]
- When the array fills, allocate a fixed-size array that’s twice as big and copy over the elements.
Time complexity of a single insertion

A single insertion:

$$T_{\text{insert}}(n) = \mathcal{O}(n)$$
Time complexity of a sequence of insertions

Hence, for a sequence of insertions:

\[ T_{\text{insert-sequence}}(m) = \sum_{i=1}^{m} O(i) \]
Time complexity of a sequence of insertions

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\[ = O \left( \sum_{i=1}^{m} i \right) \]
Time complexity of a sequence of insertions

Hence, for a sequence of insertions:

\[ T_{\text{insert-sequence}}(m) = \sum_{i=1}^{m} O(i) \]

\[ = O \left( \sum_{i=1}^{m} i \right) \]

\[ = O(1 + 2 + \cdots + (m - 1) + m) \]
Time complexity of a sequence of insertions

Hence, for a sequence of insertions:

\[ T_{\text{insert-sequence}}(m) = \sum_{i=1}^{m} \mathcal{O}(i) \]

\[ = \mathcal{O} \left( \sum_{i=1}^{m} i \right) \]

\[ = \mathcal{O}(1 + 2 + \cdots + (m - 1) + m) \]

\[ = \mathcal{O} \left( \frac{m(m + 1)}{2} \right) \]
Time complexity of a sequence of insertions

Hence, for a sequence of insertions:

\[
T_{\text{insert-sequence}}(m) = \sum_{i=1}^{m} O(i)
\]

\[
= O \left( \sum_{i=1}^{m} i \right)
\]

\[
= O \left( 1 + 2 + \cdots + (m - 1) + m \right)
\]

\[
= O \left( \frac{m(m + 1)}{2} \right)
\]

\[
= O(m^2)
\]
Amortized time complexity

Amortized time complexity considers the cost of a sequence of operations by paying attention to the state of the data structure.
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Then it apportions the time evenly among the operations.
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Then it apportions the time evenly among the operations. Amortization is about the worst case, not merely the average case.
Banker’s method: real costs vs. accounting costs

Let $c_i$ be the actual cost of the $i$th operation
Let $c'_i$ be the charged cost of the $i$th operation
Banker’s method: real costs vs. accounting costs

Let $c_i$ be the actual cost of the $i$th operation
Let $c_i'$ be the charged cost of the $i$th operation—we choose this!
Banker’s method: real costs vs. accounting costs

Let $c_i$ be the actual cost of the $i$th operation
Let $c'_i$ be the charged cost of the $i$th operation—we choose this!

If total actual cost does not exceed the total charged cost,

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} c'_i,$$

then we say that the $i$th operation has worst-case amortized time $O(c'_i)$,
Amortized time for dynamic array insertion (banker style)

Consider the \( i \)th insert operation (which results in size \( i \)):

\[
\begin{array}{c|ccccccccccc}
  i  & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]
Amortized time for dynamic array insertion (banker style)

Consider the $i$th insert operation (which results in size $i$):

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<tr>
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Let $cap_i$ be the capacity after operation $i$
Amortized time for dynamic array insertion (banker style)

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$12:6$
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(banker style)

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\[
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 i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 cap_i & 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16 & 16 \\
 c_i & 1 & 2 & 3 & 1 & 5 & 1 & 1 & 1 & 9 & 1 \\
 c'_i & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
 bal_i & 2 & 3 & 3 & 5 & 3 & 5 & 7 & 9 & 3 & 5 \\
\end{array}
\]

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Physicist’s method: potential “energy”

We define a potential function $\Phi$ on data structure states, where:

\[
\begin{align*}
\Phi(v_0) &= 0 \quad \text{starts at 0} \\
\Phi(v_t) &\geq 0 \quad \text{never goes negative}
\end{align*}
\]
Physicist’s method: potential “energy”

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$\Phi$ is akin to the balance in the banker’s method, but history-less
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$$

$$
\Phi(v_t) \geq 0 \quad \text{never goes negative}
$$

$\Phi$ is akin to the balance in the banker’s method, but history-less

We then define the amortized time of an operation:

$$
c_i' = c_i + \Phi(v_i) - \Phi(v_{i-1})
$$

$$
= c_i + \Delta \Phi(v_i)
$$
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We then define the amortized time of an operation:

$$c'_i = c_i + \Phi(v_i) - \Phi(v_{i-1})$$

$$= c_i + \Delta \Phi(v_i)$$
Potential function for dynamic arrays

We choose a potential function

\[ \Phi(v) = 2n - m , \]

where \( n \) is the size and \( m \) the capacity of \( v \).
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Let’s check \( \Phi \)’s properties:
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✓ The initial vector has no size and no capacity, so

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Let’s check \( \Phi \)’s properties:

✓ The initial vector has no size and no capacity, so
\[ \Phi(v_0) = 0 \]

The capacity is never more than twice the size, because we double when it’s full.
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where \( n \) is the size and \( m \) the capacity of \( v \).

Let’s check \( \Phi \)'s properties:

✓ The initial vector has no size and no capacity, so \( \Phi(v_0) = 0 \)

The capacity is never more than twice the size, because we double when it’s full; hence \( 2n \geq m \).
Potential function for dynamic arrays

We choose a potential function

\[ \Phi(v) = 2n - m , \]

where \( n \) is the size and \( m \) the capacity of \( v \).

Let’s check \( \Phi \)’s properties:

✓ The initial vector has no size and no capacity, so \( \Phi(v_0) = 0 \)

✓ The capacity is never more than twice the size, because we double when it’s full; hence \( 2n \geq m \); hence \( \Phi(v) = 2n - m \geq 0 \).
Amortized time for dynamic array insertion (physicist style)

Let's compute $c'_i$ for insertion. Remember that $c'_i = c_i + \Phi(v_i) - \Phi(v_{i-1})$. There are two possibilities:
Amortized time for dynamic array insertion
(physicist style)

Let’s compute $c'_i$ for insertion. Remember that

$$ c'_i = c_i + \Phi(v_i) - \Phi(v_{i-1}) $$

There are two possibilities:

- **If** $n < m$ **then** $c_i = 1$. 

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Amortized time for dynamic array insertion
(physicist style)

Let’s compute $c_i'$ for insertion. Remember that

$$c_i' = c_i + \Phi(v_i) - \Phi(v_{i-1}).$$

There are two possibilities:

If $n < m$ then $c_i = 1$. So

$$c_i' = 1 + (2(n + 1) - m) - (2n - m)$$
Amortized time for dynamic array insertion
(physicist style)

Let’s compute $c'_i$ for insertion. Remember that
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If $n = m$ then $c_i = n + 1$ (copy plus simple insert).
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If \( n = m \) then \( c_i = n + 1 \) (copy plus simple insert). So
\[
c_i' = n + 1 + (2(n + 1) - 2m) - (2n - m)
= n + 1 + (2(n + 1) - 2n) - (2n - n) \quad \text{because } n = m
\]
Amortized time for dynamic array insertion (physicist style)

Let’s compute $c'_i$ for insertion. Remember that $c'_i = c_i + \Phi(v_i) - \Phi(v_{i-1})$. There are two possibilities:

✓ If $n < m$ then $c_i = 1$. So

$$c'_i = 1 + (2(n + 1) - m) - (2n - m)$$
$$= 1 + 2 = 3$$

✓ If $n = m$ then $c_i = n + 1$ (copy plus simple insert). So

$$c'_i = n + 1 + (2(n + 1) - 2m) - (2n - m)$$
$$= n + 1 + (2(n + 1) - 2n) - (2n - n)$$
$$= 1 + 2 + n + 2n - 2n + 2n - n = 3$$
Another example: (naïve) persistent banker’s queue

A data structure is *persistent* when modifications do not destroy the previous state of the structure.
Another example: (naïve) persistent banker’s queue

A data structure is *persistent* when modifications do not destroy the previous state of the structure. (The opposite is *ephemeral*.)
Another example: (naïve) persistent banker’s queue

A data structure is *persistent* when modifications do not destroy the previous state of the structure. (The opposite is *ephemeral*.)

What if we want a persistent FIFO queue with sub-linear operations?
Take-aways

- What is *amortized time*?
- How does amortized time differ from *average time*?
- When is amortized time useful, and when might we want to avoid it?
- How can we figure out the amortized time of data structure operations?
- How does a dynamic array achieve its amortized time complexity?