The necessity of balance
The necessity of balance

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lceil \lg n \rceil$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>100</td>
<td>7</td>
</tr>
<tr>
<td>1,000</td>
<td>10</td>
</tr>
<tr>
<td>10,000</td>
<td>14</td>
</tr>
<tr>
<td>100,000</td>
<td>17</td>
</tr>
<tr>
<td>1,000,000</td>
<td>20</td>
</tr>
<tr>
<td>10,000,000</td>
<td>24</td>
</tr>
<tr>
<td>100,000,000</td>
<td>27</td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>30</td>
</tr>
</tbody>
</table>
DSSL2 tree setup

# A RandNumTree is one of:
# - node(Number, Natural, RandNumTree, RandNumTree)
# - False
defstruct node(key, size, left, right)

def size(t):
    t.size if node?(t) else 0

def new_node(k):
    node(k, 1, False, False)

def fix_size!(n):
    n.size = 1 + size(n.left) + size(n.right)

def empty?(t): t === False
Leaf insertion in DSSL2

The easy way to add elements to a tree—at the leaves:

def leaf_insert!(t, k):
    if empty?(t): new_node(k)
    elif k < t.key:
        t.left = leaf_insert!(t.left, k)
        fix_size!(t)
        t
    elif k > t.key:
        t.right = leaf_insert!(t.right, k)
        fix_size!(t)
        t
    else: t
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion
Leaf insertion

```
      7
     / \
    3   11
   / \ / \  
  1  5  9  13
 / \ / \ / \  
0  2  4  9  14
```
Leaf insertion

```
    7
   / \
  3   11
 /     /
1     9   13
|     |   |
0     8   14
```

Leaf insertion
Leaf insertion
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 — severely unbalanced (degenerate)
- 7, 3, 1, 0, 2, 5, 4, 6, 11, 9, 8, 10, 13, 12, 14 — balanced
- 7, 11, 3, 13, 9, 5, 1, 14, 12, 10, 8, 6, 4, 2, 0 — balanced

In fact, the only sequence to produce the right-branching degenerate tree is 0, …, 14

There are 21,964,800 sequences that produce the same perfectly balanced tree
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 — severely unbalanced (degenerate)
- 7, 3, 1, 0, 2, 5, 4, 6, 11, 9, 8, 10, 13, 12, 14 — balanced
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 — severely unbalanced (degenerate)
- 7, 3, 1, 0, 2, 5, 4, 6, 11, 9, 8, 10, 13, 12, 14 — balanced
- 7, 11, 3, 13, 9, 5, 1, 14, 12, 10, 8, 6, 4, 2, 0 — balanced

In fact, the only sequence to produce the right-branching degenerate tree is 0, …, 14.
There are 21,964,800 sequences that produce the same perfectly balanced tree.
The permutation distribution

Can we characterize how sequences of insertions produce (un)balanced trees?

- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 — severely unbalanced (degenerate)
- 7, 3, 1, 0, 2, 5, 4, 6, 11, 9, 8, 10, 13, 12, 14 — balanced
- 7, 11, 3, 13, 9, 5, 1, 14, 12, 10, 8, 6, 4, 2, 0 — balanced

In fact, the only sequence to produce the right-branching degenerate tree is 0, ..., 14

There are 21,964,800 sequences that produce the same perfectly balanced tree
A random BST tends to be balanced

If you generate a tree by leaf-inserting a random permutation of its elements, it will probably be balanced

In particular, the expected length of a search path is

\[ 2 \ln n + O(1) \]
A random BST tends to be balanced

If you generate a tree by leaf-inserting a random permutation of its elements, it will probably be balanced.

In particular, the expected length of a search path is

\[ 2 \ln n + \mathcal{O}(1) \]

Unfortunately, we usually can’t do that, but we can simulate it.
A tool: tree rotations

Note that order is preserved
def rotate_right!(d):
    let b = d.left
    d.left = b.right
    b.right = d
    fix_size!(d)
    fix_size!(b)
    b

def rotate_left!(b):
    let d = b.right
    b.right = d.left
    d.left = b
    fix_size!(b)
    fix_size!(d)
    d

In DSSL2
Root insertion

Using rotations, we can insert at the root:

- To insert into an empty tree, create a new node
- To insert into a non-empty tree, if the new key is greater than the root, then root-insert (recursively) into the right subtree, then rotate left
- By symmetry, if the key belongs to the left of the old root, root insert into the left subtree and then rotate right
def root_insert!(t, k):
    if empty?(t): new_node(k)
    elif k < t.key:
        t.left = root_insert!(t.left, k)
        rotate_right!(t)
    elif k > t.key:
        t.right = root_insert!(t.right, k)
        rotate_left!(t)
    else: t
Randomized insertion

We can now build a randomized insertion function that maintains the random shape of the tree:

- Suppose we insert into a subtree of size $k$, so the result will have size $k + 1$
- If the tree were random, the new element would be the root with probability $\frac{1}{k+1}$
- So we root insert with that probability, and otherwise recursively insert into a subsubtree
def insert!(t, k):
    if empty?(t): new_node(k)
    elif random(size(t) + 1) == 0:
        root_insert!(t, k)
    elif k < t.key:
        t.left = insert!(t.left, k)
        fix_size!(t)
        t
    elif k > t.key:
        t.right = insert!(t.right, k)
        fix_size!(t)
        t
    else: t
Deletion idea

To delete a node, we join its subtrees recursively, randomly selecting which contributes the root (based on size):
def join!(t1, t2):
    if empty?(t1): t2
    elif empty?(t2): t1
    elif random(size(t1) + size(t2)) < size(t1):
        t1.right = join!(t1.right, t2)
        fix_size!(t1)
        t1
    else:
        t2.left = join!(t1, t2.left)
        fix_size!(t2)
        t2
Delete in DSSL2

```python
def delete!(t, k):
    if empty?(t): t
    elif k < t.key:
        t.left = delete!(t.left, k)
        fix_size!(t)
        t
    elif k > t.key:
        t.right = delete!(t.right, k)
        fix_size!(t)
        t
    else:
        join!(t.left, t.right)
```
Next time: guaranteed balance