

# Efficient Model Order Reduction via Multi-Node Moment Matching

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**Abstract** - The new concept of Multi-node Moment Matching (MMM) is introduced in this paper. The MMM technique simultaneously matches the moments at several nodes of a circuit using explicit moment matching around  $s=0$ . As compared to the well-known Single-point Moment Matching (SMM) techniques (such as AWE), MMM has several advantages. First, the number of moments required by MMM is significantly lower than SMM for a reduced order model of the same accuracy, which directly translates into computational efficiency. This higher computational efficiency of MMM as compared to SMM increases with the number of inputs to the circuit. Second, MMM has much better numerical stability as compared to SMM. This characteristic allows MMM to calculate an arbitrarily high order approximation of a linear system, achieving the required accuracy for systems with complex responses. Finally, MMM is highly suitable for parallel processing techniques especially for higher order approximations while SMM has to calculate the moments sequentially and cannot be adapted to parallel processing techniques.

## I. Introduction and Background

Moment matching techniques, *e.g.*, [1]-[3], are currently one of the most popular linear circuit simulation techniques. The moments of a transfer function of order  $n$  results from expanding the transfer function into a Taylor series around  $s = 0$  as given by

$$H(s) = \frac{1 + a_1s + a_2s^2 + \dots + a_ns^n}{b_0 + b_1s + b_2s^2 + \dots + b_ns^n} = m_0 + m_1s + m_2s^2 + m_3s^3 + \dots \quad (1)$$

The  $i^{\text{th}}$  moment of the transfer function  $m_i$  is the coefficient of  $s^i$  in the series expansion. To illustrate the relation between the moments, poles, and residues of the transfer function, (1) can be expressed as a partial fractions sum given by

$$H(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \dots + \frac{k_n}{s - p_n}, \quad (2)$$

where  $p_i$  is the  $i^{\text{th}}$  pole of the transfer function and  $k_i$  is the corresponding residue. By expanding each term in (2) into powers of  $s$ , the moments of  $H(s)$  can be expressed as

$$\begin{aligned} m_0 &= -\left(\frac{k_1}{p_1} + \frac{k_2}{p_2} + \dots + \frac{k_n}{p_n}\right), \\ m_1 &= -\left(\frac{k_1}{p_1^2} + \frac{k_2}{p_2^2} + \dots + \frac{k_n}{p_n^2}\right), \\ &\vdots \\ m_{2n-1} &= -\left(\frac{k_1}{p_1^{2n}} + \frac{k_2}{p_2^{2n}} + \dots + \frac{k_n}{p_n^{2n}}\right). \end{aligned} \quad (3)$$

This favorable reciprocal relation between the moments and the poles stresses the dominant poles with smaller magnitudes. These dominant poles are of most interest when evaluating the transient response. This characteristic makes the moments very popular in circuit simulation. Moreover, the moments around  $s = 0$  can be calculated very easily for tree structured and tree like interconnect in linear time

with the number of elements in the circuit [1]-[5]. Path tracing techniques for efficiently calculating the moments for tree and tree like structures were introduced in [4] and [5]. Note that tree like structures include capacitively and inductively coupled trees. Also, other techniques have been developed in [5] to extend the efficiency of path tracing techniques to circuits with few resistive and inductive loops. The overwhelming majority of interconnects in integrated circuits fall into these categories of circuits. This efficiency of calculating the moments around  $s = 0$  further increased the popularity of moment matching techniques.

Asymptotic Waveform Evaluation (AWE) [2], [3] employs moment matching by calculating the first  $2q$  moments of the transfer function around  $s = 0$  to determine the first  $q$  dominant poles and corresponding residues of the transfer function. The moments at node  $j$  are approximated by

$$\begin{aligned} m_0^j &= -\left(\frac{k_1^j}{p_1} + \frac{k_2^j}{p_2} + \dots + \frac{k_q^j}{p_q}\right), \\ m_1^j &= -\left(\frac{k_1^j}{p_1^2} + \frac{k_2^j}{p_2^2} + \dots + \frac{k_q^j}{p_q^2}\right), \\ &\vdots \\ m_{2q-1}^j &= -\left(\frac{k_1^j}{p_1^{2q}} + \frac{k_2^j}{p_2^{2q}} + \dots + \frac{k_q^j}{p_q^{2q}}\right). \end{aligned} \quad (4)$$

where the terms representing poles with magnitude larger than  $p_q$  are neglected and  $p_1 < p_2 < \dots < p_q$ . Hence, the first  $q$  most dominant poles and corresponding residues can be calculated by solving the set of  $2q$  non-linear equations with  $2q$  variables in (4). Indirect methods to solve (4) were also developed in [2] but are not explained here.

The process by which the moments are determined does not allow calculating the moments at few selected nodes of a circuit and the moments have to be calculated at all nodes since the  $i + 1$  moment at any node  $j$  depends on the  $i^{\text{th}}$  moments at all the nodes [1]-[5]. However, AWE only uses the moments at a single node at a time to calculate the response at that node. Hence, techniques such as AWE will be called here Single-point Moment Matching (SMM) techniques. These techniques improve the approximation accuracy by calculating more moments at each node. As will be shown in this paper, accuracy can be also improved by using the information in the moments at different nodes simultaneously. This new concept will be called here Multi-node Moment Matching (MMM). By exploiting the spatial information in the moments, the number of moments required to achieve a specific accuracy can be significantly reduced, improving the computational efficiency. As will be described later, the reduction in the number of moments by using MMM instead of SMM increases with the number of inputs to the circuit. Multiple input circuits are becoming increasingly common in integrated circuits with the increasing importance of analyzing interconnects with capacitive and inductive coupling. Also, important structures such as the power distribution network are typically multi-input circuits.

Another major problem with SMM techniques is the inherent numerical instability with higher order approximations. The

set of equations in (4) is very sensitive to numerical errors with high  $q$  due to the high powers of the poles involved. Even with a moderate disparity in the pole values, higher moments very quickly contain no information about larger magnitude poles due to truncation errors. For that reason SMM techniques are limited to less than 8-10 poles [4]-[6]. With complicated integrated circuits, increasing inductance effects, higher inductive and capacitive coupling, and higher operating frequencies, approximations with orders higher than can be achieved by SMM become necessary.

Techniques such as Complex Frequency Hopping (CFH) have been proposed to determine higher number of poles [7]-[8]. CFH calculates the moments around several frequency points instead of only around  $s = 0$ . A different set of poles is emphasized around the selected frequency point in each set of moments, allowing the calculation of high number of poles. However, calculating the moments around  $s \neq 0$  cannot use path tracing techniques and is much more complicated than calculating the moments around  $s = 0$  especially when inductive and capacitive coupling are present. Also, determining the set of points around which the moments are calculated is a non-trivial task. Another set of techniques that are becoming increasingly popular are techniques based on Krylov subspaces, e.g., [9]-[14]. These techniques implicitly matches the moments of the circuit by using a different set of vectors that have the same span of the moment vectors but are much more numerically stable. Very high approximation orders can be achieved by using these techniques. However, these techniques have significantly higher complexity than AWE. Techniques based on the Arnoldi algorithm, e.g., [9]-[11], improve the numerical stability by finding a set of orthogonal vectors with the same information as the moments. Finding this orthogonal set of vectors has a complexity proportional to  $q^2n$  where  $n$  is the total number of states in the circuit [9]-[11]. It is currently a common belief that explicit moment matching around  $s=0$  cannot be used to calculate high order approximations.

This paper introduces the MMM technique capable of calculating arbitrarily high order approximations with high efficiency using *explicit* matching of the moments *around*  $s = 0$ . The algorithm has a complexity proportional to  $qn$  for tree and tree-like structures and actually has significantly better performance than SMM techniques. Hence, MMM has better numerical stability as well as higher computational efficiency as compared to AWE, unlike the techniques described above which trade of computational efficiency for numerical stability. The rest of the paper is organized as follows. The special case of a single input multiple output systems will be considered in section II. The general case of multiple input multiple output systems are considered in section III. The use of dummy inputs to arbitrarily improve the numerical characteristics of the MMM technique is described in section IV. Conclusions are given in section V. Finally, the relation between the residues of the transfer functions and the eigen vectors of the system matrix is derived in the Appendix.

## II. MMM for a Single Input Multiple Output System

A formal method for simultaneously matching  $q + 1$  moments at  $q$  points of a circuit is described in this section. Consider a linear circuit with  $n$  state variables (independent capacitor voltages and inductor currents). Out of the  $n$  state variables,  $q$  variables are selected to represent the circuit. Selection criteria are discussed later in the section. These state variables are denoted  $x_1, x_2, \dots, x_q$  or  $\mathbf{x}$  in vector notation. The  $qx1$  vector  $\mathbf{m}_i$  includes the  $i^{\text{th}}$  moments of the state vector  $\mathbf{x}$  due to a unit impulse input. A reduced order state space system of order  $q$  is to be determined which approximate the original circuit by simultaneously matching the moments of the selected  $q$  state variables  $\mathbf{x}$ . This system is given by

$$\dot{\mathbf{x}}_{qx1} = \mathbf{A}_{qxq} \mathbf{x}_{qx1} + \mathbf{b}_{qx1} u_{1x1}, \quad (5)$$

where  $\mathbf{A}$  is a  $q \times q$  system matrix,  $\mathbf{b}$  is a  $q \times 1$  input vector, and  $u$  is the single input to the circuit. The state variables of the reduced order system have a one to one correspondence with the selected state variables from the original circuit  $x_1, x_2, \dots, x_q$ . To approximate the selected state variables of the original circuit, a reduced order system with the first  $q + 1$  moments of  $\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_q$  is to be determined. Therefore, (5) can be expressed in the frequency domain as

$$s \cdot [\mathbf{m}_0 + \mathbf{m}_1 s + \mathbf{m}_2 s^2 + \dots + \mathbf{m}_q s^q + \dots] = \mathbf{A}[\mathbf{m}_0 + \mathbf{m}_1 s + \mathbf{m}_2 s^2 + \dots + \mathbf{m}_q s^q + \dots] + \mathbf{b} \cdot 1, \quad (6)$$

where  $u(s)$  is equal to 1 for a unit impulse input. By comparing the coefficients of equal powers of  $s$  in both sides, the following relations have to be satisfied

$$\begin{aligned} \mathbf{b} &= -\mathbf{A}\mathbf{m}_0, \\ \mathbf{m}_0 &= \mathbf{A}\mathbf{m}_1, \\ \mathbf{m}_1 &= \mathbf{A}\mathbf{m}_2, \\ &\vdots \\ \mathbf{m}_{q-1} &= \mathbf{A}\mathbf{m}_q. \end{aligned} \quad (7)$$

Excluding the first equation, the equations above can be put in matrix form as

$$\mathbf{A}[\mathbf{m}_1 \mathbf{m}_2 \dots \mathbf{m}_q] = [\mathbf{m}_0 \mathbf{m}_1 \dots \mathbf{m}_{q-1}], \quad (8)$$

Hence, a reduced order system can be determined which has the same  $q + 1$  moments of the original state variables using the following two equations

$$\mathbf{A} = \Lambda_1 \Lambda_2^{-1}, \quad (9)$$

$$\mathbf{b} = -\mathbf{A}\mathbf{m}_0, \quad (10)$$

where  $\Lambda_1$  and  $\Lambda_2$  are two  $qxq$  matrices given by

$$\Lambda_1 = [\mathbf{m}_0 \mathbf{m}_1 \dots \mathbf{m}_{q-1}], \quad (11)$$

$$\Lambda_2 = [\mathbf{m}_1 \mathbf{m}_2 \dots \mathbf{m}_q]. \quad (12)$$

The eigen values of  $\mathbf{A}$ ,  $p_1, p_2, \dots, p_q$ , are the reduced order common set of poles of the circuit. The residues of the transfer function between the input and any state variable  $x_j$  in the original circuit can be calculated by solving the set of linear equations given by

$$\begin{aligned} m_0^j &= -\left(\frac{k_1^j}{p_1} + \frac{k_2^j}{p_2} + \dots + \frac{k_q^j}{p_q}\right), \\ m_1^j &= -\left(\frac{k_1^j}{p_1^2} + \frac{k_2^j}{p_2^2} + \dots + \frac{k_q^j}{p_q^2}\right), \\ &\vdots \\ m_{q-1}^j &= -\left(\frac{k_1^j}{p_1^q} + \frac{k_2^j}{p_2^q} + \dots + \frac{k_q^j}{p_q^q}\right). \end{aligned} \quad (13)$$

Note that the state variable  $x_j$  can be any variable in the original circuit and does not have to be one of the  $q$  variables used in calculating the reduced order system. Once the poles and residues of the transfer functions at the nodes of interest are determined, the response of the circuit at these nodes to an arbitrary input can be calculated using simple Laplace transform techniques.

Determining a reduced order system of order  $q$  using the MMM technique requires  $q + 1$  moments ( $\mathbf{m}_0 - \mathbf{m}_q$ ). The number of moments required for any SMM technique to determine a reduced order system of order  $q$  is  $2q$  [1]-[8] as discussed in section I. Hence,

almost half the number of moments are required by MMM as compared to SMM to calculate an approximation of the same order. The reason MMM uses less moments is that it exploits the fact that there is a common set of poles at all the nodes of a circuit. By only considering a single node at a time, SMM requires  $2q$  moments to solve for  $2q$  variables ( $q$  poles and their residues). However, by adding more nodes, the number of variables does not increase by  $2q$  for each extra node. Since the  $q$  poles are common to all the nodes, adding an extra node only adds  $q$  new variables for  $q$  residues at the extra node. Hence, the number of variables when simultaneously considering  $q$  nodes is  $q$  poles and  $q^2$  residues and MMM needs only to match  $q(q+1)$  moments which are  $q+1$  moments at  $q$  nodes.

Using  $q+1$  moments instead of  $2q$  moments does not reduce the accuracy of an approximation calculated based on MMM as compared to SMM. Although there is no formal proof for this argument, this trait can be illustrated in several ways. Intuitively, although MMM uses less moments, the reduced order system in (5) represents the original circuit more closely by matching the circuit characteristics at more than one point rather than at a single point. Alternatively, in the special case when  $q = n$ , MMM exactly matches the circuit using  $n+1$  moments while SMM requires  $2n$  moments to exactly match the circuit. For example, consider the simple circuit in Fig. 1. MMM replicates the exact response at all the nodes using eleven moments as shown in Fig. 2 for the output node while SMM requires 20 moments to replicate the responses (SMM breaks in this example due to numerical errors when using a sixteen significant decimal digit arithmetic). Also, note in Fig. 3 that for an eighth order approximation MMM is actually more accurate than SMM and for a fourth order approximation, MMM is as accurate as SMM. Another way to illustrate the similar accuracy of a  $q^{\text{th}}$  order approximation using MMM and SMM is to note that both techniques implicitly solve the same set of equations in (4) and (13) to find  $q$  poles and  $q$  residues. The only difference is that MMM solves the system of equations more efficiently by exploiting the fact that there is a common set of poles at all the nodes.

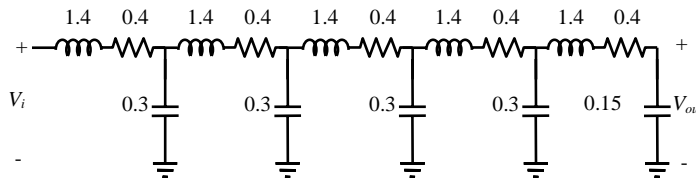


Fig. 1. Simple RLC circuit.

The selection of the  $q$  variables used in calculating an MMM reduced order model is critical for the accuracy of the MMM approximation. For example, if two state variable with exactly the same moments are selected, two identical rows appear in  $\Lambda_2$  and the matrix cannot be inverted as required by (9). The choice of two state variables with close moments can also cause loss of accuracy. Hence, as a rule of thumb, the variables should be selected with as far moments as possible. A way to achieve this objective is to sort the state variables in terms of the first moment and select the variables at equidistant steps starting with the variable with the smallest moment and ending with the variable with the largest moment. Also, if inductors are present in the circuit, both inductor currents and capacitor voltages should be selected as state variables when calculating the reduced order model. Intuitively, by choosing the variables according to these criteria, the circuit is more comprehensively sampled and hence is more accurately approximated by the reduced order model. Also, in many cases the  $q$  variables are implied by the physical nature of the circuit.

Another interesting observation is that MMM does not use the higher half of the moments which SMM uses for the same approximation order  $q$ . Thus, the maximum power of the poles in the

system of equations solved by MMM is almost half that of SMM. As described in section I, the high powers of the poles are the primary reason for truncation errors and numerical instability in SMM techniques. Hence, MMM is numerically more stable than SMM. This fact is evident in Fig. 2. Although the numerical advantage of MMM being able to reach twice the order of SMM is not huge, it is shown in section III that both the numerical and computational advantages of MMM over SMM increase in direct proportion to the number of inputs to the circuit. This fact is exploited in section IV by using dummy inputs to arbitrarily improve the numerical characteristics of MMM, allowing the calculation of approximations with practically any required number of poles.

Finally, moment shifting can readily be applied in MMM and was shown in [6] to improve the accuracy of moment matching approximations by eliminating the inaccuracy effects of larger magnitude poles on the dominant poles. A reduced order system of the form in (5) can be calculated with a moment shifting of  $sh$  by using equation (9) with

$$\Lambda_1 = [\mathbf{m}_{sh} \mathbf{m}_{sh+1} \cdots \mathbf{m}_{sh+q-1}], \quad (14)$$

$$\Lambda_2 = [\mathbf{m}_{sh+1} \mathbf{m}_{sh+2} \cdots \mathbf{m}_{sh+q}], \quad (15)$$

and

$$\mathbf{b} = -\mathbf{A}^{sh+1} \mathbf{m}_{sh}. \quad (16)$$

A shift between 1-3 usually results in significant accuracy improvement of the reduced order model for RLC circuits.

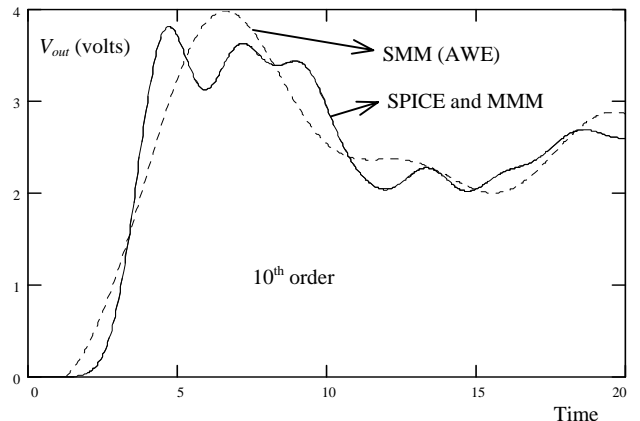
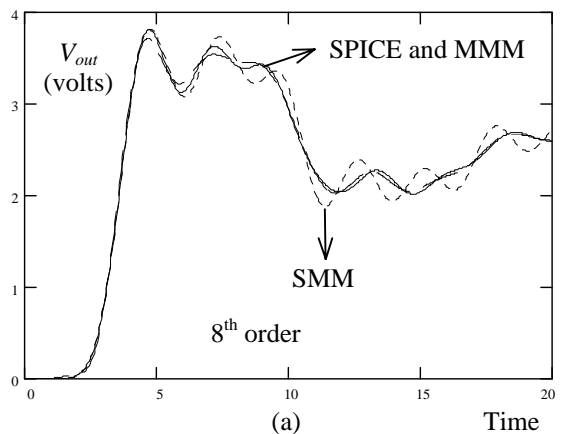


Fig. 2. Tenth order MMM and SMM approximations as compared to SPICE for the output voltage of the circuit in Fig. 1. MMM requires eleven moments and is exact while SMM requires 20 moments and breaks because of numerical errors.



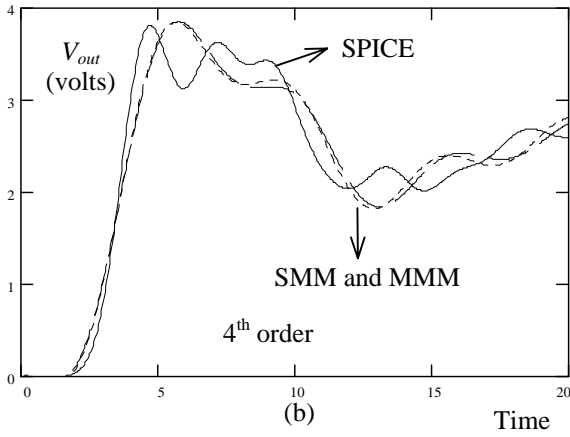


Fig. 3. Eighth and fourth order MMM and SMM approximations as compared to SPICE for the output voltage of the circuit in Fig. 1.

### III. MMM for a Multiple Input Multiple Output System

For a circuit with  $I$  inputs, a reduced order system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (17)$$

is used as a  $q$  order approximation of the original circuit, where  $\mathbf{B}$  is a  $q \times I$  input matrix and  $\mathbf{u}$  is an  $I \times 1$  vector including the inputs to the circuit. This reduced order system can be expressed as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2 + \dots + \mathbf{b}_I u_I, \quad (18)$$

where  $u_1 - u_I$  are the  $I$  inputs and  $\mathbf{b}_1 - \mathbf{b}_I$  are the corresponding columns of  $\mathbf{B}$ . Unlike the single input case, there are  $I$  different sets of moment vectors that can be calculated for the multiple input system in (18). A different set of moment vectors results from setting one input as a unit impulse while all the other inputs are set to zeros. The set of moment vectors corresponding to the input  $u_k$  are denoted  $\mathbf{m}_{0,k}, \mathbf{m}_{1,k}, \dots$ , where  $\mathbf{m}_{i,k}$  includes the  $i^{\text{th}}$  moments of the  $q$  state variables in  $\mathbf{x}$  selected to represent the circuit. Note that these moments are the moments of the transfer functions between input  $k$  and the state variables of the circuit since a circuit with  $I$  inputs has  $I$  different transfer functions at each node  $j$  as described by

$$x_j(s) = H_1^j(s)u_1(s) + H_2^j(s)u_2(s) + \dots + H_I^j(s)u_I(s). \quad (19)$$

Following the same procedure as in the single input case, the following relations can be shown to hold when matching the moments of the reduced order system to the moments of the  $q$  variables  $x_1, x_2, \dots, x_q$  of the original circuit

$$\begin{aligned} \mathbf{b}_k &= -\mathbf{A}\mathbf{m}_{0,k}, \\ \mathbf{m}_{i,k} &= \mathbf{A}\mathbf{m}_{i+1,k}, \end{aligned} \quad (20)$$

for any  $i = 0, 1, 2, \dots$  and any  $k = 1, 2, \dots, I$ . Any  $q + I$  moment vectors can be used to completely determine an approximation (18) of order  $q$  since (18) has  $(q+I)q$  variables ( $q^2$  elements in  $\mathbf{A}$  and  $qI$  elements in  $\mathbf{b}_1 - \mathbf{b}_I$ ). The only condition is that at least  $\mathbf{m}_{0,k}$  needs to be determined for all the inputs ( $k = 1 \dots I$ ) to determine  $\mathbf{b}_k$  as given by (20). Hence,  $\mathbf{A}$  can be determined using (9) with several different alternatives for  $\Lambda_1$  and  $\Lambda_2$ . One interesting choice is

$$\Lambda_1 = \begin{bmatrix} \mathbf{m}_{0,1} \mathbf{m}_{1,1} \dots \mathbf{m}_{q-1,1} & \mathbf{m}_{0,2} \mathbf{m}_{1,2} \dots \mathbf{m}_{q-1,2} & \dots & \mathbf{m}_{0,I} \mathbf{m}_{1,I} \dots \mathbf{m}_{q-1,I} \end{bmatrix}, \quad (21)$$

$$\Lambda_2 = \begin{bmatrix} \mathbf{m}_{1,1} \mathbf{m}_{2,1} \dots \mathbf{m}_{q-1,1} & \mathbf{m}_{1,2} \mathbf{m}_{2,2} \dots \mathbf{m}_{q-1,2} & \dots & \mathbf{m}_{1,I} \mathbf{m}_{2,I} \dots \mathbf{m}_{q-1,I} \end{bmatrix}, \quad (22)$$

which uses the first  $q/I + 1$  moments corresponding to each input  $u_1 - u_I$ . The reduced order system in (18) calculated using (21) and (22) implicitly solves the set of equations given by

$$\begin{aligned} \mathbf{m}_{0,k} &= -\left(\frac{1}{p_1} \mathbf{k}_{1,k} + \frac{1}{p_2} \mathbf{k}_{2,k} + \dots + \frac{1}{p_q} \mathbf{k}_{q,k}\right), \\ \mathbf{m}_{1,k} &= -\left(\frac{1}{p_1^2} \mathbf{k}_{1,k} + \frac{1}{p_2^2} \mathbf{k}_{2,k} + \dots + \frac{1}{p_q^2} \mathbf{k}_{q,k}\right), \\ &\vdots \\ \mathbf{m}_{\frac{q}{I},k} &= -\left(\frac{1}{p_1^{\frac{q}{I}+1}} \mathbf{k}_{1,k} + \frac{1}{p_2^{\frac{q}{I}+1}} \mathbf{k}_{2,k} + \dots + \frac{1}{p_q^{\frac{q}{I}+1}} \mathbf{k}_{q,k}\right), \end{aligned} \quad (23)$$

for  $k = 1 \dots I$  where  $\mathbf{k}_{i,k}$  is a vector including the residues of the  $i^{\text{th}}$  pole  $p_i$  at the  $q$  variables  $x_1, x_2, \dots, x_q$ . At a first glance, solving these equations may seem impossible since there are  $(q+I)q$  equations when matching  $(q+I)$  moments at  $q$  nodes while the number of variables involved is  $q^2 I + q$ . These variables are  $q$  common poles and  $q^2 I$  residues since at each of the  $q$  nodes there are  $I$  different transfer functions (each with  $q$  residues) as given by (19). However, as shown in the Appendix, for any linear system, the residues of the transfer functions due to different inputs are not completely independent since the residue vectors are related to the eigen vectors of  $\mathbf{A}$  by

$$\begin{aligned} \mathbf{k}_{1,k} &= a_{1,k} \mathbf{v}_1, \\ \mathbf{k}_{2,k} &= a_{2,k} \mathbf{v}_2, \\ &\vdots \\ \mathbf{k}_{q,k} &= a_{q,k} \mathbf{v}_q, \end{aligned} \quad (24)$$

for  $k = 1 \dots I$  where  $\mathbf{v}_i$  is the eigen vector of  $\mathbf{A}$  corresponding to the  $i^{\text{th}}$  pole  $p_i$  and  $a_{1,k} - a_{q,k}$  are a set of  $q$  constants unique to each input  $u_k$ . This set of constants can be expressed in vector notation as  $\alpha_k$  and is determined from

$$\alpha_k = \mathbf{T}^{-1} \mathbf{b}_k, \quad (25)$$

where

$$\mathbf{T} = [\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_q] \quad (26)$$

as explained in the appendix. Note that the above relations hold for any linear system and therefore are also valid for the original circuit of dimension  $n$  (or when  $q = n$ ). Hence, the system of equations in (23) is equivalent to

$$\begin{aligned} \mathbf{m}_{0,k} &= -\left(\frac{\alpha_{1,k}}{p_1} \mathbf{v}_1 + \frac{\alpha_{2,k}}{p_2} \mathbf{v}_2 + \dots + \frac{\alpha_{q,k}}{p_q} \mathbf{v}_q\right), \\ \mathbf{m}_{1,k} &= -\left(\frac{\alpha_{1,k}}{p_1^2} \mathbf{v}_1 + \frac{\alpha_{2,k}}{p_2^2} \mathbf{v}_2 + \dots + \frac{\alpha_{q,k}}{p_q^2} \mathbf{v}_q\right), \\ &\vdots \\ \mathbf{m}_{\frac{q}{I},k} &= -\left(\frac{\alpha_{1,k}}{p_1^{\frac{q}{I}+1}} \mathbf{v}_1 + \frac{\alpha_{2,k}}{p_2^{\frac{q}{I}+1}} \mathbf{v}_2 + \dots + \frac{\alpha_{q,k}}{p_q^{\frac{q}{I}+1}} \mathbf{v}_q\right), \end{aligned} \quad (27)$$

for  $k = 1 \dots I$ . The system of equations in (27) has  $(q+I)q$  independent variables and  $(q+I)q$  equations and hence can be solved. The residues of any state variable that is part of the  $q$  variables  $x_1, x_2, \dots, x_q$  selected to represent the original circuit can be calculated directly

from (24). For a state variable  $x_j$  in the original circuit outside the set  $x_1, x_2, \dots, x_q$ , the residues can be found by solving the following set of  $q$  linear equations

$$\begin{aligned} m_{0,k}^j &= -\left(\frac{\alpha_{1,k}}{p_1} v_1^j + \frac{\alpha_{2,k}}{p_2} v_2^j + \dots + \frac{\alpha_{q,k}}{p_q} v_q^j\right), \\ m_{1,k}^j &= -\left(\frac{\alpha_{1,k}}{p_1} v_1^j + \frac{\alpha_{2,k}}{p_2} v_2^j + \dots + \frac{\alpha_{q,k}}{p_q} v_q^j\right), \\ &\vdots \\ m_{\frac{q-1}{I},k}^j &= -\left(\frac{\alpha_{1,k}}{p_1^{\frac{q}{I}}} v_1^j + \frac{\alpha_{2,k}}{p_2^{\frac{q}{I}}} v_2^j + \dots + \frac{\alpha_{q,k}}{p_q^{\frac{q}{I}}} v_q^j\right), \end{aligned} \quad (28)$$

for  $k = 1 \dots I$  where  $v_i^j$  is the component of the  $i^{\text{th}}$  eigen vector at node  $j$ . Note that the poles  $p_1 - p_q$  and the constants  $\alpha_{1,k} - \alpha_{q,k}$  are known in (28) since they can be determined from the reduced order system and are common to all nodes. The residues at node  $j$  can then be determined using the  $j^{\text{th}}$  component of (24) given by

$$k_{i,k}^j = \alpha_{i,k} v_i^j, \quad (29)$$

for  $i = 1, 2, \dots, q$  and  $k = 1, 2, \dots, I$ .

The entire procedure of determining a multiple input reduced order system can be summarized as follows. The reduced order system itself can be determined from (20) and (21). The set of poles  $p_1 - p_q$  are the eigen values of  $\mathbf{A}$  which are common to all the nodes in the circuit. The set of constants  $\alpha_{1,k} - \alpha_{q,k}$  which are also common to all the nodes in the circuit are determined from (25) and (26). Finally, the residues at any node  $j$  can be determined either directly from (24) if the node belongs to the reduced order system or from (28) and (29) for any node in the original circuit outside the set of nodes used in the reduced order system.

The number of moments required to determine a reduced order model of order  $q$  for a circuit with  $I$  inputs using MMM is  $q + I$  moments. The number of moments required for SMM is at least  $q + qI$  moments to determine  $q$  common poles and  $q$  residues for each of the  $I$  inputs. Note that the savings in the number of moments required by MMM as compared to SMM increases dramatically with the number of inputs to the circuit especially for large  $q$ . This savings is again because MMM exploits the fact that the poles  $p_1 - p_q$  and the constants  $\alpha_{1,k} - \alpha_{q,k}$  are common to all the nodes in the circuit. By considering only one node, SMM techniques cannot exploit this fact.

For example, consider the three capacitively coupled RC transmission lines shown in Fig. 4. A third order MMM approximation is used to simulate the circuit. The outputs of the three transmission lines are chosen as the three variables representing the circuit. The first two moments,  $\mathbf{m}_{0,k}$  and  $\mathbf{m}_{1,k}$ , are calculated for each of the three inputs, i.e.,  $k = 1, 2$ , and 3. The approximation is then calculated using

$$\Lambda_1 = [\mathbf{m}_{0,1} \mathbf{m}_{0,2} \mathbf{m}_{0,3}], \quad \Lambda_2 = [\mathbf{m}_{1,1} \mathbf{m}_{1,2} \mathbf{m}_{1,3}]. \quad (30)$$

The 3<sup>rd</sup> order MMM approximation is compared to SPICE in Fig. 5 and accurately approximates the transient response of the circuit for different input switching conditions. Note that an MMM approximation does not need to be recalculated for different inputs. The total number of moments calculated by MMM is six. A 3<sup>rd</sup> order SMM approximation requires at least 12 moments. Moreover,  $\mathbf{m}_{0,k}$  can usually be trivially calculated for most VLSI interconnects. Hence, the actual number of moments that are calculated by MMM is three as compared to nine by SMM, which illustrates the efficiency of MMM as compared to SMM. This efficiency is even higher for circuits with higher number of inputs.

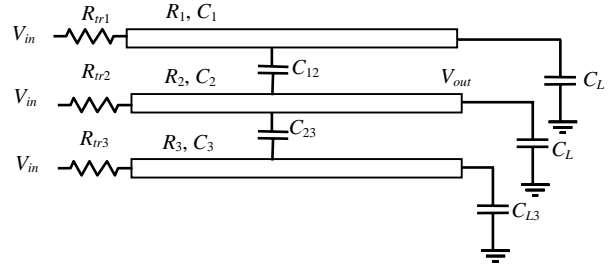


Fig. 4. Three capacitively coupled RC transmission lines.

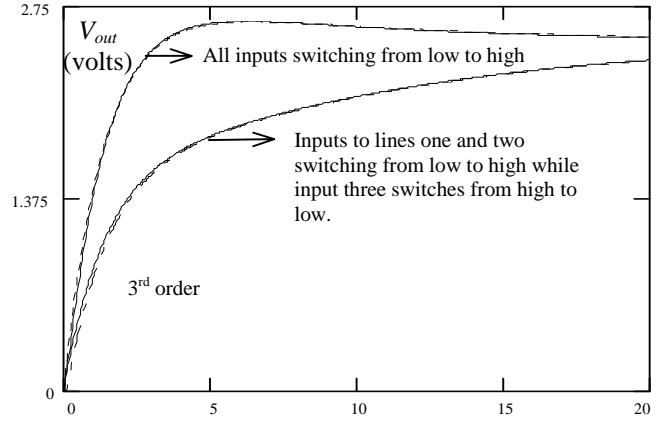


Fig. 5. MMM as compared to SPICE for  $V_{out}$  in the circuit shown in Fig. 4. The circuit element values are:  $R_1=0.8$ ,  $C_1=1$ ,  $R_{ir1}=0.5$ ,  $C_{L1}=1$ ,  $R_2=0.5$ ,  $C_2=0.5$ ,  $R_{ir2}=0.5$ ,  $C_{L2}=0.5$ ,  $R_3=0.2$ ,  $C_3=2$ ,  $R_{ir3}=1$ ,  $C_{L3}=1$ ,  $C_{12}=2$ , and  $C_{23}=3$ . A third order MMM approximation is used. SPICE simulations are shown with solid lines while MMM is shown with dotted lines.

Another very important observation is that the numerical stability of a  $q$  order approximation based on MMM increases as the number of inputs to the circuit increase. This behavior can be explained by observing that the maximum power of the poles in (20)-(29) is  $(q/I+1)$ . Hence, for a given  $q$ , an increase in  $I$  would result in a direct improvement in the numerical stability of the reduced order model. Note that the maximum power of the poles in an SMM approximation of order  $q$  is  $2q$  independent of the number of inputs to the circuit. For example, for a 40<sup>th</sup> order approximation, the maximum power of the poles is 80 in SMM and only five in the case of MMM with ten inputs to the circuit, which is a huge difference. While typical circuits have less than ten inputs, dummy inputs can be arbitrarily introduced as discussed in the next section, allowing fine control of the numerical characteristics of an MMM approximation. In the limit, the maximum power of the poles in an MMM approximation of any order can be limited to two in the special case when  $q = I$ . The minimum power of the poles in an MMM approximation can also be controlled by employing moment shifting [6] as discussed in section II, allowing an even greater control of the numerical characteristics of an MMM approximation. Equation (9) can be used to determine a reduced order multiple input system with a moment shifting of  $sh$  by using

$$\Lambda_1 = \left[ \mathbf{m}_{sh,1} \mathbf{m}_{sh+1,1} \dots \mathbf{m}_{sh+\frac{q}{I}-1,1} \mathbf{m}_{sh,2} \mathbf{m}_{sh+1,2} \dots \mathbf{m}_{sh+\frac{q}{I}-1,2} \dots \mathbf{m}_{sh,I} \mathbf{m}_{sh+1,I} \dots \mathbf{m}_{sh+\frac{q}{I}-1,I} \right], \quad (31)$$

$$\Lambda_2 = \left[ \mathbf{m}_{sh+1,1} \mathbf{m}_{sh+2,1} \dots \mathbf{m}_{sh+\frac{q}{I},1} \mathbf{m}_{sh+1,2} \mathbf{m}_{sh+2,2} \dots \mathbf{m}_{sh+\frac{q}{I},2} \dots \mathbf{m}_{sh+1,I} \mathbf{m}_{sh+2,I} \dots \mathbf{m}_{sh+\frac{q}{I},I} \right], \quad (32)$$

and

$$\mathbf{b}_k = -\mathbf{A}^{sh+1} \mathbf{m}_{sh,k}, \quad (33)$$

for  $k = 1, 2, \dots, I$ .

#### IV. Controlling the Numerical Characteristics of an MMM Approximation by Using Dummy Inputs

As discussed in the previous section, the maximum power of the poles in an MMM approximation decreases as the number of inputs to the circuit increases, which improves the numerical stability of the reduced order model. In many cases, circuits with single or few inputs require high order approximations to accurately characterize the transient response of the circuit. In such cases, dummy inputs can be introduced to reduce the truncation errors in high order approximations due to the high powers of the poles involved. After the reduced order model is calculated using the circuit's original inputs and the dummy inputs, the dummy inputs can be set to zeros in the reduced order model. The added dummy inputs should satisfy two conditions. First, setting a dummy input to zero should not change the circuit structure. For example, setting a voltage source in parallel with a capacitor to zero results in short circuiting the capacitor, changing the original circuit. Second, the moment vectors due to a dummy input should be easily calculated by using path tracing techniques [4], [5] where they apply. Also, if matrix factorization is necessary for calculating the moments of the original circuit, calculating the moments due to a dummy input should not require any extra matrix factorization than that is required for calculating the moments of the original inputs. One type of input that satisfies these conditions is a voltage source in series with an inductor or a resistor in the circuit as shown in Fig. 6. The voltage source should not be in parallel with any element in the circuit and should not be between any node and the ground. The voltage source should not be in series with a capacitor. It can be easily verified that such an input satisfies the two conditions above.

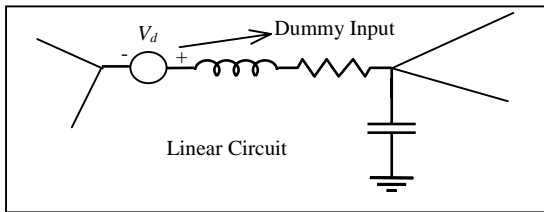


Fig. 6. A voltage source dummy input.

Dummy inputs of this type can be added until the maximum power of the poles in the MMM approximation are sufficiently low to guarantee the numerical stability of the approximation. However, to achieve the maximum linear independence between the moment vectors due to different inputs, the inputs should be selected at nodes that are physically as far as possible from each other. A technique that worked very successfully was selecting each dummy input to be orthogonal to all the previous inputs and their calculated moments. This method guarantees maximum linear independence between the moments due to different dummy inputs. The underlying idea is that the moments due to a certain input roughly represent the space of the residue vectors of the dominant poles as given by (23). Hence, by selecting the new dummy inputs orthogonal to the moments, these dummy inputs are effectively also orthogonal to the residues of the dominant poles. This characteristic means that the dominant poles will not appear in the moments of the new dummy inputs and only the higher poles will appear, allowing very high order approximations to be reached.

A simulation tool has been developed based on the MMM algorithm using the aforementioned techniques to efficiently and accurately simulate general *RLC* circuits. The tool has been used with several circuits of varying sizes and complexity to verify the accuracy and speed of the MMM method. For example, consider an underdamped *RLC* transmission line with a total resistance, inductance, and capacitance of 2, 1, and 1.5, respectively, and a single input. An underdamped transmission line is known to have one

of the most complicated responses in all single input circuits with a very poor pole separation. Applying the SMM technique at the load node using 16 significant decimal digit arithmetic, only an eighth order approximation can be reached before running into serious numerical errors. By applying MMM with nine dummy inputs (a total of ten inputs including the original input) introduced at equal distances starting from the source and ending at the load, a 40<sup>th</sup> order approximation is accurately calculated. A moment shifting of two was used with MMM. As shown in Fig. 7, the 40<sup>th</sup> order approximation accurately matches SPICE simulations and requires the calculation of 70 moments around  $s = 0$ . The maximum error in any of the poles calculated using MMM was less than 2.5% as compared to the exact poles.

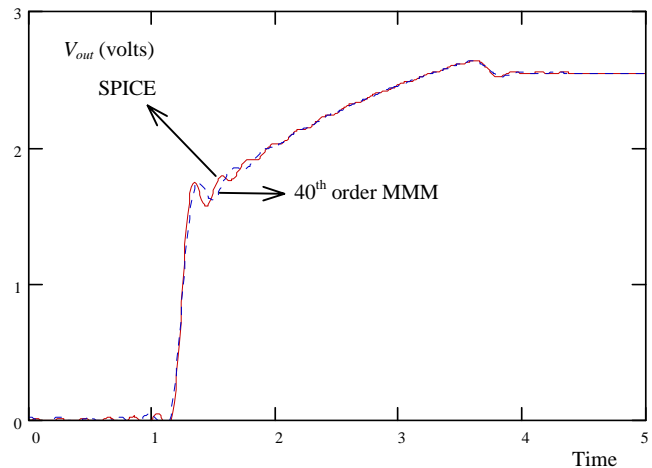


Fig. 7. An MMM approximation of order 40 as compared to SPICE simulations of the output of an underdamped *RLC* transmission line.

The seventy moments used in the above example are composed of seven moments for each of the ten inputs. These ten sets of seven moments do not depend on each other and can be calculated in parallel. As for SMM or Krylov subspace methods [9]-[14], the moment vectors (or the equivalent vectors in the case of Krylov-based methods) have to be calculated sequentially and thus, parallel programming techniques cannot be employed. This advantage of the MMM technique can be significant especially when higher order approximations are required which will usually involve a large number of moment sets due to different inputs with each sets having only few moments to limit the maximum power of the poles in the approximation.

Consider also the case of a two identical coupled *RLC* transmission lines. A 40<sup>th</sup> order approximation is calculated using MMM with ten inputs (eight dummy and two original inputs) and a moment shifting of two. The approximation matches accurately SPICE simulations at the end of line two for several input switching conditions as shown in Fig. 8. Seventy moments are used again which represents no increase as compared to the single transmission line example given above. A sixteen significant decimal digit arithmetic was used.

Note that transmission lines are used here not because of their simplicity but rather because transmission lines involve some of the most complicated signals in linear circuits and because the presented results can be easily verified. MMM can be used with any linear circuit for which the moments can be calculated.

The MMM technique has been tested on a large number of industrial circuits with great success, resulting in orders of magnitude improvements in speed over existing tools. Table 1 summarizes the results of applying MMM on one large industrial circuit. This circuit is a full grid clock distribution network with a significant amount of coupling. Note that this unoptimized version of MMM requires only few seconds to accurately simulate the circuit on a 1.7 GHz PC with

128 Mb of RAM. The results were consistent across a wide range of approximations and approximations as high as 60 were achieved. For this specific circuit, an approximation of order 30 or higher showed no difference in the result, *i.e.*, a 30<sup>th</sup> order is enough for accurately simulating this circuit. A 30<sup>th</sup>, 40<sup>th</sup>, and 50<sup>th</sup> order approximations are shown in Fig. 9 with no visible difference.

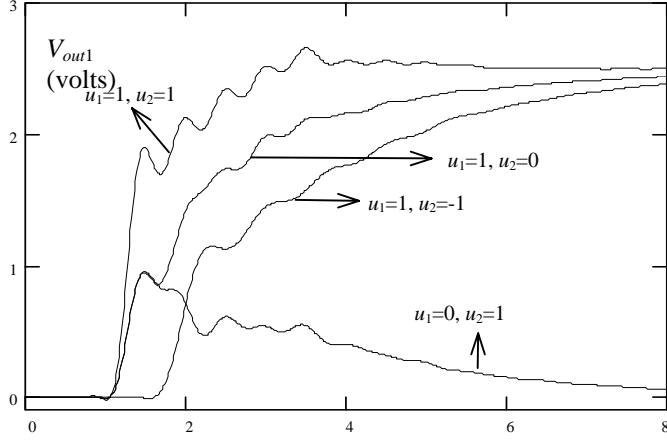


Fig. 8. An MMM approximation of order 40 as compared to SPICE simulations at the output of line one of two coupled identical *RLC* transmission lines. The total resistance, inductance, and capacitance of the two lines are 2, 1, and 1.5, respectively, and the coupling capacitance between the two lines is 1. The parameters  $u_1$  and  $u_2$  takes the values 1, 0, or -1 indicating that the corresponding line is switching from low to high, not switching, or switching from high to low, respectively. The MMM approximation is indistinguishable from SPICE simulations.

Table 1. Large circuit simulation using MMM.

Circuit type:	Full grid clock distribution network
No. of Capacitors:	21982
No. of Resistors:	15432
No. of Inductors:	3860
No. of Mutual Inductance Coupling:	7340
No. of Nodes:	19177
SPICE simulation time	SPICE couldn't finish this circuit. One quarter of this circuit took 6 days on SPICE.
MMM simulation time	10-25 seconds depending on the approximation order used

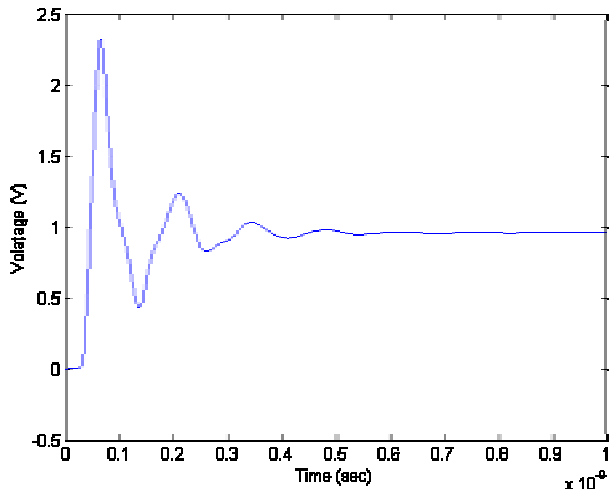


Fig. 9. An MMM approximation of order 30 as compared to SPICE simulations for the circuit described in Table 1.

## V. Conclusions

The new concept of Multi-node Moment Matching was introduced in this paper and explicitly matches the moments around  $s = 0$ . As compared to SMM techniques, MMM has the following advantages. First, the number of moments required by MMM as compared to SMM is significantly lower for a reduced order model of the same accuracy. This higher computational efficiency of MMM as compared to SMM increases with the number of inputs to the circuit. Second, MMM has much better numerical stability as compared to SMM, allowing MMM to calculate approximations of much higher orders than SMM can achieve. Finally, MMM is highly suitable for parallel processing techniques especially for higher order approximations while SMM has to calculate the moments sequentially and cannot be adapted to parallel processing techniques.

## Appendix: Relation between the Residues and the Eigen Vectors of the System Matrix

Consider a single input system with a dimension  $q$  of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u. \quad (34)$$

By taking the Laplace transform of (34) and assuming a unit impulse,  $\mathbf{x}(s)$  can be expressed as

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}. \quad (35)$$

This expression can be expanded into powers of  $s$  as

$$\mathbf{x}(s) = -[\mathbf{A}^{-1} + \mathbf{A}^{-2}s + \mathbf{A}^{-3}s^2 + \dots]\mathbf{b}. \quad (36)$$

Hence, the moment vectors of  $\mathbf{x}(s)$  are given by

$$\mathbf{m}_i = -\mathbf{A}^{-(i+1)}\mathbf{b}. \quad (37)$$

The vector  $\mathbf{b}$  can be expressed as a linear combination of the eigen vectors of  $\mathbf{A}$  as given by

$$\mathbf{b} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_q\mathbf{v}_q, \quad (38)$$

where the constants  $\alpha_1$ - $\alpha_q$  (expressed in vector notation as  $\boldsymbol{\alpha}$ ) can be determined from

$$\boldsymbol{\alpha} = \mathbf{T}^{-1}\mathbf{b}, \quad (39)$$

with

$$\mathbf{T} = [\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_q]. \quad (40)$$

Substituting (38) into (36) and using the well known relation  $f(\mathbf{A})\mathbf{v}_i = f(p_i)\mathbf{v}_i$  where  $p_i$  is the eigen value of  $\mathbf{A}$  which corresponds to the eigen vector  $\mathbf{v}_i$ , the following relation results

$$\mathbf{m}_i = -\left[ \frac{\alpha_1}{p_1^{i+1}}\mathbf{v}_1 + \frac{\alpha_2}{p_2^{i+1}}\mathbf{v}_2 + \dots + \frac{\alpha_q}{p_q^{i+1}}\mathbf{v}_q \right], \quad (41)$$

However, the relation between the moments and the residues in (3) can be expressed in vector form as

$$\mathbf{m}_i = -\left[ \frac{1}{p_1^{i+1}}\mathbf{k}_1 + \frac{1}{p_2^{i+1}}\mathbf{k}_2 + \dots + \frac{1}{p_q^{i+1}}\mathbf{k}_q \right], \quad (42)$$

where  $\mathbf{k}_i$  is the vector containing the residues corresponding to the pole  $p_i$  at all the nodes of the circuit. By comparing (41) and (42), the residue vectors are just the scaled eigen vectors given by

$$\mathbf{k}_i = \alpha_i \mathbf{v}_i. \quad (43)$$

While the significance of this relation is not so obvious for single output systems, this relation has significant consequences for multiple input systems. Consider a linear system with  $I$  inputs of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2 + \dots + \mathbf{b}_I u_I. \quad (44)$$

The same process used for a single input system can be used on each input with all other inputs equal to zero, resulting in the following relations

$$\begin{aligned} \mathbf{k}_{1,k} &= a_{1,k} \mathbf{v}_1, \\ \mathbf{k}_{2,k} &= a_{2,k} \mathbf{v}_2, \\ &\vdots \\ \mathbf{k}_{q,k} &= a_{q,k} \mathbf{v}_q, \end{aligned} \quad (45)$$

for  $k = 1 \dots I$  where  $\mathbf{k}_{i,k}$  is a vector including the residues of the  $i^{\text{th}}$  pole  $p_i$  due to a unit impulse at  $u_k$  and  $\alpha_{1,k} - \alpha_{q,k}$  are a set of  $q$  constants unique to each input which can be determined from

$$\boldsymbol{\alpha}_k = \mathbf{T}^{-1} \mathbf{b}_k. \quad (46)$$

Hence, the residue vectors corresponding to a certain pole  $p_i$  due to different inputs are not completely independent but are actually just scaled versions of the same vector  $\mathbf{v}_i$ . This fact reduces the number of unknowns in the residue vectors from  $q^2 I$  to  $q^2 + qI$  which is a potentially much smaller number for large  $q$  and  $I$ . Finally, note that this relation holds for any linear system including the linear system representing the whole circuit of order  $n$ . Hence, by nature, the model order reduction technique used by MMM preserves this characteristic of the original circuit. Note also that a reduced order model determined based on MMM has a common set of poles at all the nodes represented by the eigen values of  $\mathbf{A}$  and hence, MMM also preserves this characteristic of the original circuit.

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