Network Game with a Probabilistic Description of User Types

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CDC 2004, Atlantis, Bahamas
December 17, 2004
Outline

- Previous Work
- Problem Formulation
  - Complete Information
  - Partially Incomplete Information
  - Totally Incomplete Information
- Single User
- Two Users
- Multiple Users
- Conclusions and Extensions
Previous Work

➢ Başar and Srikant (2002a, 2002b)
Hierarchical Stackelberg Network Game Model
Solution under Uniform Pricing
Asymptotic Behavior Analysis

➢ Shen and Başar (2004)
Solution under Differentiated Pricing
Comparison with Uniform Pricing and Asymptotic Behavior Analysis


Two-Level Hierarchical Network Game

ISP
\[ \max_{\{p_i\}_{i=1}^I} R \]

User 1
\[ \max_{x_1} F_1 \]

User I
\[ \max_{x_I} F_I \]

\( I \)-player noncooperative game

Leader
Stackelberg game
Followers
Problem Formulation

➢ Single Internet Service Provider (ISP)
➢ Set of users: $\mathcal{I} = \{i : 1 \leq i \leq I\}$
➢ Flow vector: $\vec{x} = (x_1, \cdots, x_I)^T \in \Omega$
➢ Price vector: $\vec{p} = (p_1, \cdots, p_I)^T$
➢ User $i$’s net utility: $f_i(x_i) + g(\vec{x}) - p_i x_i$
➢ ISP’s profit: $\vec{p}^T \vec{x}$

➢ Classification of games based on the information structure:
   1. Complete information
   2. Partially incomplete information
   3. Totally incomplete information
Complete Information

➢ The utility function of each user is common knowledge for all the users and the ISP.

➢ Given $\vec{p}$, a Nash equilibrium is $\vec{x}(\vec{p}) = (x_1(\vec{p}), \cdots, x_I(\vec{p}))^T$, such that for User $i$,

$$x_i(\vec{p}) = \arg \max_{x_i} f_i(x_i) + g(x_i, x_{-i}(\vec{p})) - p_i x_i,$$

where $x_{-i} = \{x_j\}_{j \neq i}$.

➢ There exists a unique Nash equilibrium. See Başar and Srikant (2002a).

➢ The optimal price vector for the ISP is:

$$\vec{p}^* = \arg \max_{\vec{p}:\vec{p} \geq \theta} \sum_{i \in I} p_i x_i(\vec{p}).$$

Partially Incomplete Information

➢ The utility function of each user is common knowledge for all the users, but not for the ISP.

➢ The distribution of the users’ types, \( \vec{w} = (w_1, \cdots, w_I)^T \), that determine \( f_i \)'s (\( g \) is deterministic), is known to the ISP.

➢ Given \( \vec{p} \), for each fixed \( \vec{w} \), the unique Nash equilibrium is \( x^\vec{w} (\vec{p}) = (x_1^\vec{w} (\vec{p}), \cdots, x_I^\vec{w} (\vec{p}))^T \), such that for User \( i \),

\[
x_i^\vec{w} (\vec{p}) = \arg \max_{x_i} f_i^{w_i} (x_i) + g(x_i, x_{-i}^\vec{w} (\vec{p})) - p_i x_i.
\]

➢ The optimal price vector for the ISP is:

\[
\vec{p}^* = \arg \max_{\vec{p}: \vec{p} \geq \theta} \sum_{i \in I} p_i E[ x_i^\vec{w} (\vec{p})].
\]
Totally Incomplete Information

➢ User $i$ has private information $w_i$ and knows the distribution of $w_{-i} = \{w_j\}_{j \neq i}$.

➢ The distribution of $\vec{w} = (w_1, \cdots, w_I)^T$ is known to the ISP.

➢ Given $\vec{p}$, the pure strategy Bayesian equilibrium, if it exists, is $x_i^{w_i}(\vec{p})$ for User $i$, such that

$$x_i^{w_i}(\vec{p}) = \arg \max_{x_i} f_i^{w_i}(x_i) + E_{w_{-i}}[g(x_i, x_{-i}^{w_i}(\vec{p}))] - p_i x_i.$$ 

➢ The optimal price vector for the ISP is:

$$\vec{p}^* = \arg \max_{\vec{p} : \vec{p} \geq \theta} \sum_{i \in I} p_i E_{\vec{w}}[x_i^{w_i}(\vec{p})].$$
Single User

➢ No Classification for Incomplete Information

➢ Complete Information

➢ Incomplete Information: Two-point Distribution

➢ Incomplete Information: Continuous Distribution — A Numerical Example

➢ Comparison of Complete Information and Incomplete Information
Complete Information

➢ User: $\max_x f(x) + g(x) - px = > p = f'(x(p)) + g'(x(p))$

➢ ISP: $\max_p px(p)$

➢ Graphical illustration of the solution
Incomplete Information — Two-point Distribution

- User: $x^i(p) = \arg \max_x f^i(x) + g(x) - px$ w.p. $q_i$, $i = 1, 2$
- ISP: $p^* = \arg \max_p p[q_1 x^1(p) + q_2 x^2(p)]$
- Complete information: $p^{i*} = \arg \max_p p x^i(p)$, $i = 1, 2$
- Graphical illustration of the solution and comparison
Incomplete Information — Uniform Distribution

- \( f^{\omega'}(x) = w(1 - c^{-1}x), \quad g'(x) = -ac^{-1}x \)
- \( w \) uniformly distributed over \([0, b]\)
- Numerical example: \( a = b = c = 1 \)
- Profit loss of 12.8\% for the ISP due to incomplete information
Two Users

» Complete Information and Partially Incomplete Information — Unique Nash Equilibrium

» Totally Incomplete Information — Pure Strategy Bayesian Equilibrium

» Quadratic Utility Functions
  » Complete Information
  » Partially Incomplete Information
  » Totally Incomplete Information
  » Numerical Results

» Comparison of the Three Classes of Games
Pure Strategy Bayesian Equilibrium

➢ Special case: independent users $i = 1, 2$, $f_i(x_i) = f^k_i(x_i)$ w.p. $q_k$, $k = 1, 2$

➢ Pure strategy Bayesian equilibrium: $x_1^k(p^\ast)$ and $x_2^k(p^\ast)$, $k = 1, 2$, solve

$$\max_{x_1^k} \left\{ f^k_1(x_1^k) + q_1 g(x_1^k, x_2^1(p^\ast)) + q_2 g(x_1^k, x_2^2(p^\ast)) - p_1 x_1^k \right\},$$

$$\max_{x_2^k} \left\{ f^k_2(x_2^k) + q_1 g(x_1^1(p^\ast), x_2^k) + q_2 g(x_1^2(p^\ast), x_2^k) - p_2 x_2^k \right\}$$

➢ Sufficient condition for the existence, uniqueness and stability

1. apply the techniques in Li and Başar (1987)

2. convergence by the Banach contraction mapping theorem

S. Li and T. Başar (1987), “Distributed algorithms for the computation of noncooperative equilibria,”

Quadratic Utility Functions

- $f_w(x_i) = w_i(1 - c^{-1}x_i), \ i = 1, 2$
- $\nabla g(x) = -ac^{-1}(x_1 + x_2)(1, 1)^T$
- independent users: $i = 1, 2, \ w_i = w^k$ w.p. $q_k, k = 1, 2$

- Analytical results for the three types of games

- Numerical results:
  $a = c = 1, \ w^1 = 2, \ w^2 = 1, \ q_1 = q_2 = 0.5$
Numerical Results and Comparison of Games

<table>
<thead>
<tr>
<th>$(w_1, w_2)$</th>
<th>Game</th>
<th>Optimal Price Vector</th>
<th>Optimal Flow Vector</th>
<th>ISP’s Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 2)$</td>
<td>C</td>
<td>$(1, 1)$</td>
<td>$(0.25, 0.25)$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>$(0.3, 0.3)$</td>
<td>$(0.4250, 0.4250)$</td>
<td>0.2550</td>
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<tr>
<td></td>
<td>T</td>
<td>$(0.7, 0.7)$</td>
<td>$(0.3647, 0.3647)$</td>
<td>0.5106</td>
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<tr>
<td>$(2, 1)$</td>
<td>C</td>
<td>$(1, 0.5)$</td>
<td>$(0.3, 0.1)$</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>$(0.3, 0.3)$</td>
<td>$(0.5400, 0.0800)$</td>
<td>0.1860</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>$(0.7, 0.7)$</td>
<td>$(0.3647, 0.0471)$</td>
<td>0.2882</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>C</td>
<td>$(0.5, 1)$</td>
<td>$(0.1, 0.3)$</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>$(0.3, 0.3)$</td>
<td>$(0.0800, 0.5400)$</td>
<td>0.1860</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>$(0.7, 0.7)$</td>
<td>$(0.0471, 0.3647)$</td>
<td>0.2882</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>C</td>
<td>$(0.5, 0.5)$</td>
<td>$(0.1667, 0.1667)$</td>
<td>0.1667</td>
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<tr>
<td></td>
<td>P</td>
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<td>$(0.2333, 0.2333)$</td>
<td>0.1400</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>$(0.7, 0.7)$</td>
<td>$(0.0471, 0.0471)$</td>
<td>0.0659</td>
</tr>
</tbody>
</table>

Expected Profit: C — 0.3417  P — 0.1918  T — 0.2882
Multiple Users

- Quadratic Utility Functions
  - Complete Information
  - Partially Incomplete Information
  - Totally Incomplete Information
  - Numerical Results

- Comparison of the Three Classes of Games
Quadratic Utility Functions

➢ $f_i^w (x_i) = w_i (1 - x_i), \frac{\partial g(\bar{x})}{\partial x_i} = - \sum_{j=1}^{I} x_j$

Independent users: $1 \leq i \leq I$

➢ Complete Information: Nash equilibrium — extension of differentiated pricing in Shen and Başar (2004)

➢ Partially Incomplete Information: Nash equilibrium — extension of uniform pricing in Shen and Başar (2004)

➢ Totally Incomplete Information: analytical results for the special case with four user types

Numerical Results and Comparison of Games

Numerical results:

\[ w^t = t \text{ w.p. } q_t = \frac{1}{4}, t \in \{1, 2, 3, 4\}; I = 40, 10 \text{ for each type} \]

<table>
<thead>
<tr>
<th>( i, w_i )</th>
<th>C</th>
<th>P</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i, w_i )</td>
<td>( p_i^* )</td>
<td>( x_i(p_i^*) )</td>
<td>( p^* )</td>
</tr>
<tr>
<td>1–10, 4</td>
<td>2</td>
<td>0.1341</td>
<td>1.99</td>
</tr>
<tr>
<td>11–20, 3</td>
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<td>0.0122</td>
<td>1.99</td>
</tr>
<tr>
<td>21–30, 2</td>
<td>( \geq 0.5366 )</td>
<td>0</td>
<td>1.99</td>
</tr>
<tr>
<td>31–40, 1</td>
<td>( \geq 0 )</td>
<td>0</td>
<td>1.99</td>
</tr>
</tbody>
</table>

| \( n \) | 20 | 10 | 10 |
| \( \sum x_j \) | 1.4634 | 1.4357 | 1.3559 |
| \( \sum p_j x_j \) | 2.8659 | 2.8571 | 2.7119 |
Partially? Totally?

➢ Partially Incomplete Information vs Totally Incomplete Information

Number of Users

Optimal Price

Expected Profit
Conclusions

➢ Formulation of the three classes of games

➢ Complete Information vs Incomplete Information:
  Complete Information — ISP and less aggressive users
  Incomplete Information — more aggressive users

➢ Partially Incomplete Information vs Totally Incomplete Information for the ISP:
  Partially Incomplete Information — with a large number of users
  Totally Incomplete Information — with a small number of users

Extensions

➢ General utility functions; Multiple ISPs; . . .

End of the Talk