Differentiated Internet Pricing Using a Hierarchical Network Game Model

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Outline

» General Network [BS’02]

» Complete Solution for a Special Single Link Network
  » Uniform Price (UniPri) [BS’02]
  » Differentiated Prices (DiffPri)

» Comparison of the Two Pricing Schemes

» General Single Link in a Many-User Regime

» Conclusions and Extensions

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➤ General Network
➤ Complete Solution for a Special Single Link Network
  ➤ Uniform Price (UniPri)
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➤ Comparison of the Two Pricing Schemes
➤ General Single Link in a Many-User Regime
➤ Conclusions and Extensions
Problem Formulation

➢ Single Internet Service Provider (ISP)
➢ Set of users, $\mathcal{I} = \{1, \ldots, I\}$; flow of user $i$, $x_i$, $i \in \mathcal{I}$
➢ Set of links, $\mathcal{L} = \{1, \ldots, L\}$; capacity of link $l$, $c_l$, $l \in \mathcal{L}$
➢ Set of Links $x_i$ traverses, $\mathcal{L}_i \subseteq \mathcal{L}$
➢ Unit price charged to user $i$ for using link $l$, $p_{li}$, $l \in \mathcal{L}_i$
➢ Net utility of user $i$, $(\bar{x}_l = \sum_{i:l \in \mathcal{L}_i} x_i; w_i, k_i, v_i$: positive scalars)

$$F_i = w_i \log(1 + k_i x_i) - \sum_{l \in \mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - v_i x_i \sum_{l \in \mathcal{L}_i} p_{li}$$

➢ Revenue of the ISP,

$$R = \sum_{l \in \mathcal{L}} \sum_{i:l \in \mathcal{L}_i} p_{li} x_i = \sum_{i \in \mathcal{I}} x_i \sum_{l \in \mathcal{L}_i} p_{li}$$
Two-Level Hierarchical Network Game

ISP
\[
\max_{\{p_{li}\}} R
\]

Leader

Stackelberg game

User 1
\[
\max_{x_1} F_1
\]

Followers

User \( I \)
\[
\max_{x_I} F_I
\]

\( I \)-player noncooperative game
Existence of a Unique Nash Equilibrium

➢ Suppose that prices are given and fixed.

➢ Add to $F_i$ the quantity not related to $x_i$, [BS’02]

$$
\sum_{j \neq i} w_j \log(1 + k_j x_j) - \sum_{l \notin \mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - \sum_{j \neq i} v_j x_j \sum_{l \in \mathcal{L}_j} p_{lj}.
$$

➢ Obtain an equivalent noncooperative game where all the users have a common objective function (strictly concave),

$$
F = \sum_{i \in \mathcal{I}} w_i \log(1 + k_i x_i) - \sum_{i \in \mathcal{I}} v_i x_i \sum_{l \in \mathcal{L}_i} p_{li} - \sum_{l \in \mathcal{L}} \frac{1}{c_l - \bar{x}_l}.
$$

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Special Single Link Network

➢ Single link network with a capacity $n$ shared by $n$ users

➢ Net utility of user $i$, ($\bar{x} := \sum_{j=1}^{n} x_j$)

$$F_i = w_i \log(1 + x_i) - \frac{1}{n - \bar{x}} - p_i x_i, \quad i \in N := \{1, \cdots, n\}$$

➢ Uniform Price (UniPri): $p_i = p$ (complete solution by [BS'02])

➢ Differentiated Prices (DiffPri)

➢ Notations:

$$x_{av} := \frac{\bar{x}}{n}; \quad \bar{w} := \sum_{j=1}^{n} w_j, \quad w_{av} := \frac{\bar{w}}{n}; \quad \bar{v}^{\frac{1}{2}} := \sum_{j=1}^{n} \sqrt{w_j}, \quad v_{av}^{\frac{1}{2}} := \frac{\bar{v}^{\frac{1}{2}}}{n}$$

---

Positive Solution for UniPri

\[ x_{av-u}^* = 1 - \frac{2}{1 + (n^2 w_{av})^{\frac{1}{3}}} , \uparrow \]

\[ d_u^* = \frac{1}{n - nx_{av-u}^*} = \frac{1 + (n^2 w_{av})^{\frac{1}{3}}}{2n} , \downarrow \]

\[ x_{i-u}^* = \frac{w_i}{w_{av}} (x_{av-u}^* + 1) - 1, \quad i \in N, \uparrow \]

\[ p_u^* = \frac{w_{av}}{2} (1 + (n^2 w_{av})^{-\frac{1}{3}}) - \frac{1}{4n^2} (1 + (n^2 w_{av})^{\frac{1}{3}})^2 , \]

\[ r_u^* = p_u^* x_{av-u}^* = \frac{w_{av}}{2} - \frac{3}{4n^2} (n^2 w_{av})^{\frac{2}{3}} + \frac{1}{4n^2} , \]

if and only if

\[ w_i > \frac{2(n^2 w_{av})^{\frac{2}{3}} + 2n^2 w_{av}}{4n^2} , \forall i \in N \]
Positive Solution for DiffPri

\[ x_{av-d}^* = 1 - \frac{2}{1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}, \uparrow \]

\[ d_d^* = \frac{1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{2n}, \downarrow \]

\[ x_{i-d}^* = \frac{\sqrt{w_i}}{v_{av}^{\frac{1}{2}}}(x_{av-d}^* + 1) - 1, \ i \in N, \uparrow \]

\[ p_{i-d}^* = \sqrt{w_i} \frac{u_{av}^{\frac{1}{2}}}{2}(1 + (nv_{av}^{\frac{1}{2}})^{-\frac{2}{3}}) - \frac{1}{4n^2}(1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}})^2, \ i \in N, \]

\[ r_d^* = w_{av} - \frac{1}{2n^2}(nv_{av}^{\frac{1}{2}})^2 - \frac{3}{4n^2}(nv_{av}^{\frac{1}{2}})^{\frac{4}{3}} + \frac{1}{4n^2}, \]

if and only if

\[ w_i > \frac{2(nv_{av}^{\frac{1}{2}})^{\frac{4}{3}} + (nv_{av}^{\frac{1}{2}})^2 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{4n^2}, \forall i \in N \]
General Solution

If the necessary and sufficient condition for positive solution is not satisfied:

1. order the users such that $w_i > w_j$ only if $i < j$;

2. find the largest $n^* \leq n$ such that the condition holds for the first $n^*$ users;

3. write out the positive solution for the $n^*$-user problem;

4. obtain the solution for the $n$-user problem by appending $x_i^* = 0, i > n^*$.

[BS’02]

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Conditions for Positive Solution

\[ n^2 w_{av} \geq (nv_{av}^{\frac{1}{2}})^2 \]

Condition for UniPri \( \Rightarrow \) Condition for DiffPri

DiffPri can admit more users with relatively small \( w_i \)'s than UniPri.
Same Number of Users Admitted (1)

➢ Throughput: $x_{av-u}^* \geq x_{av-d}^*$

➢ Congestion cost: $d_u^* \geq d_d^*$

➢ Individual flows:

\[
\begin{align*}
    x_{i-u}^* &> x_{i-d}^* \quad \text{if } w_i > w_x, \\
    x_{i-u}^* &= x_{i-d}^* \quad \text{if } w_i = w_x, \\
    x_{i-u}^* &< x_{i-d}^* \quad \text{if } w_i < w_x
\end{align*}
\]

➢ Prices:

\[
\begin{align*}
    p_u^* &< p_{i-d}^* \quad \text{if } w_i > w_p, \\
    p_u^* &= p_{i-d}^* \quad \text{if } w_i = w_p, \\
    p_u^* &> p_{i-d}^* \quad \text{if } w_i < w_p
\end{align*}
\]

\[w_{\max} \geq w_x \geq w_p, w_{av} \geq (v_{av}^{1/2})^2 \geq w_{\min}\]
Same Number of Users Admitted (2)

➢ Individual utilities:

\[
\begin{align*}
F_{i-u}^* &> F_{i-d}^* \text{ if } w_i > w_F, \\
F_{i-u}^* &= F_{i-d}^* \text{ if } w_i = w_F, \\
F_{i-u}^* &< F_{i-d}^* \text{ if } w_i < w_F
\end{align*}
\]

\[w_{\text{max}} \geq w_F \geq w_p\]

➢ Revenue: \(r_u^* \leq r_d^*\)
Same Number of Users Admitted - Example

\[ n = 50 \]

with \( w_i \)'s evenly distributed around \( w_{av} = 0.001 \)

from \( 0.8775e^{-3} \) through \( 1.1225e^{-3} \)

\begin{align*}
\text{Flows} & \\
\text{Prices} & \\
\text{Net utilities} &
\end{align*}
More Users Admitted for DiffPri

Compare UniPri($n$), DiffPri($n$), and DiffPri($\hat{n}$), $n < \hat{n}$:

➢ Throughput: $x_{av-d}^* < \hat{x}_{av-d}^*$
➢ Congestion cost: $d_u^* \geq d_d^* > \hat{d}_d^*$
➢ Individual flows: $x_{i-d}^* < \hat{x}_{i-d}^*$
➢ Total revenue: $R_u^* \leq R_d^* \leq \hat{R}_d^*$
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General Single Link Network

➢ Single link network with a capacity $nc$ shared by $n$ users

➢ Net utility of user $i$, ($\bar{x} := \sum_{j=1}^{n} x_j$)

$$F_i = w_i \log(1 + k_i x_i) - \frac{1}{nc - \bar{x}} - p_i x_i, \ i \in N := \{1, \cdots, n\}$$

➢ Uniform Price (UniPri): $p_i = p$

➢ Differentiated Prices (DiffPri)

➢ Notations:

$$k_{av}^{-1} := \frac{1}{n} \sum_{j=1}^{n} \frac{1}{k_j}; \ z_{av}^{\frac{1}{2}} := \frac{1}{n} \sum_{j=1}^{n} \sqrt{\frac{w_j}{k_j}}$$
Asymptotic Solution for UniPri

\[
\alpha = \frac{2c(c + k_{av}^{-1})^2}{w_{av}k_{av}^{-1}}, \quad x_{av-u}^*(n) \sim c - \alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \uparrow
\]

\[
d^*_u(n) = \frac{1}{nc - nx_{av-u}^*(n)} \sim \alpha^{\frac{1}{3}} n^{-\frac{1}{3}}, \downarrow
\]

\[
x_{i-u}^*(n) \sim \frac{w_i}{w_{av}} (c + k_{av}^{-1}) - \frac{1}{k_i} - \frac{w_i}{w_{av}} \alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \quad i \in N, \uparrow
\]

\[
p_u^*(n) \sim \frac{w_{av}}{c + k_{av}^{-1}} + \left( \frac{2c}{k_{av}^{-1}} - 1 \right) \alpha^{\frac{2}{3}} n^{-\frac{2}{3}}, \downarrow
\]

\[
r_u^*(n) = \frac{p_u^*(n)x_{av-u}^*(n)}{c} \sim \frac{w_{av}}{c + k_{av}^{-1}} - 3\alpha^{\frac{2}{3}} n^{-\frac{2}{3}}, \uparrow
\]

if and only if

\[
w_i k_i > \frac{w_{av}}{c + k_{av}^{-1}} + \frac{2c}{k_{av}^{-1}} \alpha^{\frac{2}{3}} n^{-\frac{2}{3}}, \quad \forall i \in N
\]
Asymptotic Solution for DiffPri

\[ \beta = \frac{2c(c + k_{av}^{-1})^2}{(z_{av}^2)^2}, \quad x_{av-d}(n) \sim c - \beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \uparrow \]

\[ d_d^*(n) = \frac{1}{nc - nx_{av-d}^*(n)} \sim \beta^{-\frac{1}{3}} n^{-\frac{1}{3}}, \downarrow \]

\[ x_{i-d}^*(n) \sim \frac{\sqrt{w_i k_i}}{z_{av}^{\frac{1}{2}}}(c + k_{av}^{-1}) - \frac{1}{k_i} - \frac{\sqrt{w_i k_i}}{z_{av}^{\frac{1}{2}}} \beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \quad i \in N, \uparrow \]

\[ p_{i-d}^*(n) \sim \frac{z_{av}^{\frac{1}{2}} \sqrt{w_i k_i}}{c + k_{av}^{-1}} + \left( \frac{2c \sqrt{w_i k_i}}{z_{av}^{\frac{1}{2}}} - 1 \right) \beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \quad i \in N, \downarrow \]

\[ r_d^*(n) \sim \frac{w_{av}}{c} - \frac{(z_{av}^{\frac{1}{2}})^2}{c(c + k_{av}^{-1})} - 3\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \uparrow \]

if and only if

\[ \sqrt{w_i k_i} > \frac{z_{av}^{\frac{1}{2}}}{k_{av}^{-1} + x_{av-d}^*(n)} \sim \frac{z_{av}^{\frac{1}{2}}}{c + k_{av}^{-1}} + \frac{2c}{z_{av}^{\frac{1}{2}}} \beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \quad \forall i \in N \]
Asymptotic Comparison (1)

➢ Same conclusion: DiffPri admits more users
➢ Throughput: $\tilde{x}_{av-u}^* \cong \tilde{x}_{av-d}^*, \tilde{x}_u^* \leq \tilde{x}_d^*$
➢ Congestion cost: $\tilde{d}_u^* \geq \tilde{d}_d^*$
➢ Individual flows:

\[
\begin{align*}
\tilde{x}_{i-u}^* > \tilde{x}_{i-d}^* & \text{ if } w_i k_i > \tilde{w}k, \\
\tilde{x}_{i-u}^* = \tilde{x}_{i-d}^* & \text{ if } w_i k_i = \tilde{w}k, \\
\tilde{x}_{i-u}^* < \tilde{x}_{i-d}^* & \text{ if } w_i k_i < \tilde{w}k
\end{align*}
\]

➢ Prices:

\[
\begin{align*}
\tilde{p}_u^* < \tilde{p}_{i-d}^* & \text{ if } w_i k_i > \tilde{w}k, \\
\tilde{p}_u^* = \tilde{p}_{i-d}^* & \text{ if } w_i k_i = \tilde{w}k, \\
\tilde{p}_u^* > \tilde{p}_{i-d}^* & \text{ if } w_i k_i < \tilde{w}k
\end{align*}
\]
Asymptotic Comparison (2)

➢ Individual utilities:

\[
\begin{align*}
\tilde{F}_{i-u}^* &> \tilde{F}_{i-d}^* \quad \text{if } w_i k_i > \tilde{w}k_F, \\
\tilde{F}_{i-u}^* &= \tilde{F}_{i-d}^* \quad \text{if } w_i k_i = \tilde{w}k_F, \\
\tilde{F}_{i-u}^* &< \tilde{F}_{i-d}^* \quad \text{if } w_i k_i < \tilde{w}k_F
\end{align*}
\]

\[
(w_i k_i)_{\max} \geq \tilde{w}k_F \geq \tilde{w}k := (w_{av}/z_{av}^{1/2})^2 \geq (w_i k_i)_{\min}
\]

➢ Revenue: \(\tilde{r}_u^* \leq \tilde{r}_d^*\)
How DiffPri Affects Users

DiffPri for all users: total flow $\uparrow$, congestion cost $\downarrow$

- **DiffPri**
  - flows $\uparrow$, prices $\downarrow$, net utilities $\uparrow$
- **DiffPri**
  - flows $\downarrow$, prices $\uparrow$, net utilities $\uparrow$
- **DiffPri**
  - flows $\downarrow$, prices $\uparrow$, net utilities $\downarrow$

Users
- $w_i k_i \leq \tilde{w} k$
- $\tilde{w} k < w_i k_i \leq \tilde{w} k_F$
- $\tilde{w} k_F < w_i k_i$
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Conclusions

Price differentiation leads to a more egalitarian resource distribution at fairer prices:

➢ more users admitted

➢ higher total flow, alleviated congestion

➢ beneficial to the ISP: improved revenue

➢ beneficial to users with relatively small utility parameters: reduced prices, increased flows and utilities

➢ disadvantageous to other users: decreased utilities

ISP tends to have more users admitted (UniPri or DiffPri):

➢ increased throughput and flows, reduced congestion, decreased prices and improved revenue

➢ incentive for the ISP to increase the capacity
Extensions

➢ Linear network [BS’02A] and other general networks
➢ Incomplete information
➢ Multiple ISPs


_________ End of the Talk _________