“Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge, it is deficient to the extent it assumes other features to be common knowledge, such as one player’s probability assessment about another’s preferences or information.

“I forsee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analysis of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality.”

Basic Mechanism Design Question: How should an economic system be designed so that selfish agent behavior leads to good outcomes?
Informal Thesis: *approximate optimality* is

- more *descriptive* of mechanisms in practice than exact optimality,
- *prescribes* solutions to problems where exact optimality has not, and
- more *conclusive* about salient characteristics of good mechanisms.
A Gambler’s Stopping Game:

- sequence of $n$ games,
- prize of game $i$ is distributed from $F_i$,
- prior-knowledge of distributions.

On day $i$, gambler plays game $i$:

- realizes prize $v_i \sim F_i$,
- chooses to keep prize and stop, or
- discard prize and continue.
Example 1: Gambler’s Stopping Game

A Gambler’s *Stopping Game*:

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On day \( i \), gambler plays game \( i \):

- **realizes** prize \( v_i \sim F_i \),
- chooses to keep prize and **stop**, or
- discard prize and **continue**.

**Question:** How should our gambler play?
Optimal Strategy:

• threshold $t_i$ for stopping with $i$th prize.

• solve with “backwards induction”.
Optimal Strategy:

- threshold $t_i$ for stopping with $i$th prize.
- solve with “backwards induction”.

Discussion:

- **Complicated**: $n$ different, unrelated thresholds.
- **Inconclusive**: what are properties of good strategies?
- **Non-robust**: what if order changes? what if distribution changes?
- **Non-general**: what do we learn about variants of Stopping Game?
Threshold Strategies and Prophet Inequality

Threshold Strategy: “fix $t$, gambler takes first prize $v_i \geq t$.”

(clearly suboptimal, may not accept prize on last day!)
Threshold Strategies and Prophet Inequality

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**Theorem**: *(Prophet Inequality)* For $t$ such that $\Pr[\text{“no prize”}] = 1/2$,

$$
\mathbb{E}[\text{prize for strategy } t] \geq \frac{\mathbb{E}[\max_i v_i]}{2}.
$$

[Samuel-Cahn ’84]
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[Samuel-Cahn ’84]

**Discussion**:

- **Simple**: one number \( t \).
- **Conclusive**: trade-off “stopping early” with “never stopping”.
- **Robust**: change order? change distribution above or below \( t \)?
- **General**: same solution works for similar games: invariant of “tie-breaking rule”
0. Notation:
   - $q_i = \Pr[v_i < t]$.
   - $x = \Pr[\text{never stops}] = \prod_i q_i$.

1. Upper Bound on $E[\text{max}]$:

2. Lower Bound on $E[\text{prize}]$:

3. Choose $x = 1/2$ to prove theorem.
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E[\text{max}] \leq t + E[\max_i (v_i - t)^+] 
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**Example:** is X important in MD? competition? transfers?
- no, if mech without X is constant approx
- yes, otherwise.

- Seller can always try ad hoc improvements on approximation.
Overview

   (e.g., single-item auctions)

2. Multi-dimensional Bayesian settings.
   (e.g., multi-item auctions)

3. Prior-free settings.
Part I: Approximation for single-dimensional Bayesian mechanism design

(where agent preferences are given by a private value for service, zero for no service; preferences are drawn from a distribution)
Example 2: Single-item auction

Problem: Bayesian Single-item Auction Problem

- a single item for sale,
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Goal: seller opt. auction for \( \mathbf{F} \).
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**Question:** What is optimal auction?
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![Graph showing revenue curve](image)
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Optimal Auction Design [Myerson ’81]

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7. Cor: for iid, regular dists, optimal auction is Vickrey with monopoly reserve price $\varphi^{-1}(0)$. 
Optimal Auctions:

- *iid, regular distributions*: Vickrey with monopoly reserve price.
- *general*: sell to bidder with highest positive virtual value.
Optimal Auctions:

- \textit{iid, regular distributions}: Vickrey with monopoly reserve price.
- \textit{general}: sell to bidder with highest positive virtual value.

Discussion:

- \textit{iid, regular case}: seems unlikely in practice.
- \textit{general case}: nobody runs optimal auction (too complicated?).
Approximation with reserve prices

**Question:** when is reserve pricing a good approximation?
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**Thm:** Vickrey with reserve = *constant virtual price* with 
\[ \text{Pr\[no sale\]} = \frac{1}{2} \] is a 2-approximation. [Chawla, H, Malec, Sivan '10]
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Discussion:

- constant virtual price $\Rightarrow$ bidder-specific reserves.
- simple reserve prices natural, practical, and easy to find.
- robust posted pricing with arbitrary tie-breaking works fine too.
Anonymous Reserves

**Question:** for non-identical distributions, is *anonymous reserve* approximately optimal?

(e.g., eBay)
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**Proof:** more complicated extension of prophet inequalities.

**Discussion:**

- theorem is not tight, actual bound is in $[2, 3]$.
- justifies wide prevalence.
- approximation good for *platform design.*
Extensions

Beyond single-item auctions: *general feasibility constraints*. 
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**Thm:** for non-identical regular distributions, VCG with monopoly reserves is often a 2-approximation.  
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**Thm:** non-identical (possibly irregular) distributions, *posted pricing mechanisms* are often constant approximations.  
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Proof technique:

- optimal mechanism is a virtual surplus maximizer.

- reserve-price mechanisms are virtual surplus approximators.
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Proof technique:

- optimal mechanism is a virtual surplus maximizer.

- reserve-price mechanisms are virtual surplus approximators.

**Basic Open Question:** to what extent to simple mechanisms approximate (well understood but complex) optimal ones?

**Challenges:** non-downward-closed settings, negative virtual values.
Part II: Approximation for multi-dimensional Bayesian mechanism design

(where agent preferences are given by values for each available service, zero for no service; preferences drawn from distribution)
Example 3: unit-demand pricing

**Problem:** Bayesian Unit-Demand Pricing

- a single, unit-demand consumer.
- $n$ items for sale.
- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumer’s values for each item are drawn.

**Goal:** seller optimal *item-pricing* for $\mathbf{F}$. 
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**Question:** What is optimal pricing?
Optimal Pricing: consider distribution, feasibility constraints, incentive constraints, and solve!
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Discussion:

- little conceptual insight and
- not generally tractable.
Analogy

Challenge: approximate optimal but we do not understand it?
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**Goal:** seller opt. item-pricing for $F$.  
**Goal:** seller opt. auction for $F$.  

**APPRAOXMATION AND MECHANISM DESIGN – MAY 15, 2010**
Analogy

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Note: Same informational structure.
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Goal: seller opt. auction for \( F \).

Note: Same informational structure.
Thm: for any indep. distributions, MD-PRICING \( \leq \) SD-AUCTION.
Analogy

**Challenge:** approximate optimal but we do not understand it?

**Problem:** Bayesian Unit-demand Pricing (a.k.a., MD-PRICING)
- a single, *unit-demand* buyer,
- \( n \) items for sale, and
- a dist. \( F \) from which the consumer’s value for each item is drawn.

**Goal:** seller opt. item-pricing for \( F \).

**Problem:** Bayesian Single-item Auction (a.k.a., SD-AUCTION)
- a single item for sale,
- \( n \) buyers, and
- a dist. \( F \) from which the consumers’ values for the item are drawn.

**Goal:** seller opt. auction for \( F \).

**Note:** Same informational structure.

**Thm:** for any indep. distributions, MD-PRICING \( \leq \) SD-AUCTION.

**Thm:** a constant virtual price for MD-PRICING is 2-approx.

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**Note:** Same informational structure.

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**Thm:** a constant virtual price for MD-PRICING is 2-approx.

**Proof:** prophet inequality. [Chawla, H, Malec, Sivan ’10]
Sequential Posted Pricing: agents arrive in sequence, offer posted prices.
Multi-item Auctions

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4. Instantiation: SD-PRICING $\geq \frac{1}{\beta}$ SD-AUCTION
   (virtual surplus approximation)
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Discussion:

- **robust** to agent ordering, collusion, etc.
- **conclusive**: competition not important for approximation.
- **practical**: posted pricings widely prevalent. (e.g., eBay)
- role of randomization is crucial.  
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Open Question: identify upper bounds beyond unit-demand settings that are

- conceptually tractable and
- approximable.
Part III: Approximation for prior-free mechanism design.

(mechanisms should be good for any set of agent preferences, not just given distributional assumptions)
Prior assumption: the mechanism designer knows the distribution of agent preferences.
The problem with priors

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Where does prior come from:

- historical data
  then using prior affects incentives of earlier transactions. (e.g. Coase Conjecture)

- market analysis
  accuracy depends on market size, auctions are for small markets.
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**Question:** can we design good auctions without knowledge of prior-distribution?
Approach 1: “resource” augmentation.
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- “recruit one more bidder” is prior-free strategy.
- “bicriteria” approximation result.
- **conclusive:** competition more important than optimization.
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- “recruit one more bidder” is prior-free strategy.
- “bicriteria” approximation result.
- conclusive: competition more important than optimization.
- non-generic: e.g., for $k$-unit auctions, need $k$ additional bidders.
Special Case: $n = 1$

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- So Vickrey with two bidders \(\geq\) optimal revenue from one bidder.
Example 4: digital goods

Question: how should a profit-maximizing seller sell a digital good (n bidder, n copies of item)?
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**Bayesian Optimal Solution:** if values are iid from known distribution, post the monopoly price $\varphi^{-1}(0)$. [Myerson ’81]
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Discussion:

- optimal,
- simple, but
- not prior-free
Single-Sample Auction: (for digital goods) [Dhangwatnotai, Roughgarden, Yan ’10]

1. pick random agent $i$ as sample.

2. offer all other agents price $v_i$.

3. reject $i$. 
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Approximation via Single Sample

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Discussion:

- *prior-free*.
- *conclusive*, don’t need precise distribution, only need single sample for approximation. more samples can improve approximation factor.
- *generic*, applies to general settings.
Average-case vs Worst-case

**Note:** prior-free auction cannot be optimal in every setting.
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Average Case Approximation: \[ \exists A, \forall F \in \text{IID}, \]
\[ \mathbb{E}_{v \sim F}[A(v)] \geq \frac{\mathbb{E}_{v \sim F}[\text{OPT}_F(v)]}{\beta}. \]
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**Worst Case Approximation:** \( \exists A, \forall v, \)
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### Worst Case Approximation:
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**Notes:**
- worst-case approximation implies average-case approximation.
- \( \sup_{F \in \text{IID}} OPT_F(v) \) is *prior-free performance benchmark*.
- for digital goods, prior-free benchmark = optimal posted price revenue.
Random Sampling Auction: (for digital goods) [Goldberg, H, Wright ’01]

1. Randomly partition agents into two sets.
2. Compute optimal posted prices for each set.
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**Conjecture:** Random sampling auction is worst-case 4-approximation.

**Discussion:**

- **conclusive**, market analysis can be done “on the fly”
- **worst-case** is for $n = 2$.
- **practical**, bounds approach 1 in limit with $n$.
- **generic**, analysis extends beyond digital goods.
Prior-free results extend to limited supply, downward-closed settings, non-identical distributions, other objectives, etc.

[citations omitted]
Extensions

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Open Questions:

- non-downward-closed settings?
- multi-dimensional settings?
- beyond the revelation principle?
Conclusions:

1. Approximation is more predictive, descriptive, and conclusive than exact optimality.
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Basic Open Question: attack economic impossibility w. approximation.