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Prior-Independent Auctions for Risk-Averse Agents

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We study simple and approximately optimal auctions for agents with a particular form of risk-averse preferences. We show that, for symmetric agents, the optimal revenue (given a prior distribution over the agent preferences) can be approximated by the first-price auction (which is prior independent), and, for asymmetric agents, the optimal revenue can be approximated by an auction with simple form. These results are based on two technical methods. The first is for upper-bounding the revenue from a risk-averse agent. The second gives a payment identity for mechanisms with pay-your-bid semantics.

Categories and Subject Descriptors: J.4 [**Social and Behavioral Sciences**]: Economics

Additional Key Words and Phrases: Mechanism design, risk aversion, approximation

1. INTRODUCTION

We study optimal and approximately optimal auctions for agents with risk-averse preferences. The economics literature on this subject is largely focused on either comparative statics, i.e., is the first-price or second-price auction better when agents are risk averse, or deriving the optimal auction, e.g., using techniques from optimal control, for specific distributions of agent preferences. The former says nothing about optimality but considers realistic prior-independent auctions; the latter says nothing about realistic and prior-independent auctions. Our goal is to study approximately optimal auctions for risk-averse agents that are realistic and not dependent on assumptions on the specific form of the distribution of agent preferences. One of our main conclusions is that, while the second-price auction can be very far from optimal for risk-averse agents, the first-price auction is approximately optimal for an interesting class of risk-averse preferences.

The microeconomic treatment of risk aversion in auction theory suggests that the form of the optimal auction is very dependent on precise modeling details of the preferences of agents, see, e.g., Maskin and Riley [1984] and Matthews [1984]. The resulting auctions are unrealistic because of their reliance on the prior assumption and because they are complex [cf. Wilson 1987]. Approximation can address both issues. There may be a class of mechanisms that is simple, natural, and much less dependent on exact properties of the distribution. As an example of this agenda for risk neutral agents, Hartline and Roughgarden [2009] showed that for a large class of distributional assumptions the second-price auction with a reserve is a constant approximation to the optimal single-item auction. This implies that the only information about the distribution of preferences that is necessary for

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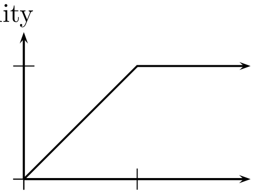
a good approximation is a single number, i.e., a good reserve price. Often from this sort of "simple versus optimal" result it is possible to do away with the reserve price entirely. Dhangwatnotai et al. [2010] and Roughgarden et al. [2012] show that simple and natural mechanisms are approximately optimal quite broadly. We extend this agenda to auction theory for risk-averse agents.

The least controversial approach for modeling risk-averse agent preferences is to assume agents are endowed with a concave function that maps their wealth to a utility. This introduces a non-linearity into the incentive constraints of the agents which in most cases makes auction design analytically intractable. We therefore restrict attention to a very specific form of risk aversion that is both computationally and analytically tractable: utility functions that are linear up to a given capacity and then flat. We will refer to these as *capacitated* utility functions. Importantly, an agent with such a utility function will not trade off a higher probability of winning for a lower price when the utility from such a lower price is greater than her capacity. While capacitated utility functions are unrealistic, they form a basis for general concave utility functions. In our analyses we will endow the benchmark optimal auction with knowledge of the agents' value distribution and capacity; however, some of the mechanisms we design to approximate this benchmark will be oblivious to them.



1: The capacitated utility function, $u_C(z) = \min(z, C)$.

As an illustrative example, consider the problem of maximizing welfare by a single-item auction when agents have known capacitated utility functions (but unknown values). Recall that for risk-neutral agents the second-price auction is welfare-optimal as the payments are transfers from the agents to the mechanism and cancel from the objective welfare which is thus equal to value of the winner. (The auctioneer is assumed to have linear utility.) For agents with capacitated utility, the second-price auction can be far from optimal. For instance, when the difference between the highest and second highest bid is much larger than the capacity then the excess value (beyond the capacity) that is received by the winner does not translate to extra utility because it is truncated at the capacity. Instead, a variant of the second-price auction, where the highest bidder wins and is charged the maximum of the second highest bid and her bid less her capacity, obtains the optimal welfare. Unfortunately, this auction is parameterized by the form of the utility function of the agents. There is, however, an auction, not dependent on specific knowledge of the utility functions or prior distribution, that is also welfare optimal: If the agents values are drawn i.i.d. from a common prior distribution then the first-price auction is welfare-optimal. To see this: (a) standard analyses show that at equilibrium the highest-valued agent wins, and (b) no agent will shade her bid more than her capacity as she receives no increased utility from such a lower payment but her probability of winning strictly decreases.

Our main goal is to duplicate the above observation for the objective of revenue. It is easy to see that the gap between the optimal revenues for risk-neutral and capacitated agents can be of the same order as the gap between the optimal welfare and the optimal revenue (which can be unbounded). When the capacities are small the revenue of the welfare-optimal auction for capacitated utilities is close to its welfare (the winners utility is at most her capacity). When capacities are infinite or very large then the risk-neutral optimal revenue is close to the capacitated optimal revenue (the capacities are not binding). One of our main technical results shows that even for mid-range capacities one of these two mechanisms that are optimal at the extremes is close to optimal.

As a first step towards understanding profit maximization for capacitated agents, we characterize the optimal auction for agents with capacitated utility functions. We then give a "simple versus optimal" result showing that either the revenue-optimal auction for risk-neutral agents or the above welfare-optimal auction for capacitated agents is a good approximation to the revenue-optimal auction for capacitated agents. In symmetric settings, the Bulow-Klemperer [1996] Theorem implies that with enough competition (and mild distributional assumptions) welfare-optimal auctions are approximately revenue-optimal. The first-price auction is welfare-optimal and prior-independent; therefore we conclude that it is approximately revenue-optimal for capacitated agents in symmetric settings. The asymmetric setting, however, is much harder — even for risk-neutral bidders no similar result is known.

Our "simple versus optimal" result (which holds for both symmetric and asymmetric settings) comes from an upper bound on the expected payment of an agent in terms of her allocation rule [cf. Myerson 1981]. This upper bound is the most technical result in the paper; the difficulties that must be overcome by our analysis are exemplified by the following observations. First, unlike in risk-neutral mechanism design, Bayes-Nash equilibrium does not imply monotonicity of allocation rules. There are mechanisms where an agent with a high value would prefer less overall probability of service than she would have obtained if she had a lower value (Example 3.4 in Section 3). Second, even in the case where the capacity is higher than the maximum possible value of any agent, the optimal mechanism for risk-averse agents can generally obtain more revenue than the optimal mechanism for risk-neutral agents (Example 4.4 in Section 4). This may be surprising because, in such a case, the revenue-optimal mechanism for risk-neutral agents would give any agent a wealth that is within the linear part of her utility function. Finally, while our upper bound on risk-averse payments implies that this relative improvement is bounded by a factor of two for large capacities, it can be arbitrarily large for small capacities (Example 4.3 in Section 4).

It is natural to conjecture that the first-price auction will continue to perform nearly optimally well beyond our simple model (capacitated utility) of risk-averse preferences. It is a relatively straightforward calculation to see that for a large class of risk-averse utility functions from the literature [e.g., Matthews 1984] the first-price auction is approximately optimal at extremal risk parameters (risk-neutral or extremely risk-averse). We leave to future work the extension of our analysis to mid-range risk parameters for these other families of risk-averse utility functions.

It is significant and deliberate that our main theorem is about the first-price auction which is well known to not have a truthtelling equilibrium. Our goal is a prior-independent mechanism. In particular, we would like our mechanism to be parameterized neither by the distribution on agent preference nor by the capacity that governs the agents utility function. While it is standard in mechanism design and analysis to invoke the *revelation principle* [cf. Myerson 1981] and restrict attention to auctions with truthtelling as equilibrium, this principle cannot be applied in prior-independent auction design. An auction with good equilibrium can be implemented by one with truthtelling as an equilibrium if the agent strategies can be simulated by the auction. In a Bayesian environment, agent strategies are parameterized by the prior distribution and therefore the suggested revelation mechanism is not generally prior independent.

Risk Aversion, Universal Truthfulness, and Truthfulness in Expectation. Our results have an important implication on a prevailing and questionable perspective that is explicit and implicit broadly in the field of algorithmic mechanism design. Two standard solution concepts from algorithmic mechanism design are "universal truthfulness" and "truthfulness in expectation." A mechanism is universally truthful if an agent's optimal (and dominant) strategy is to reveal her values for the various outcomes of the mechanism regardless of the reports of other agents or random coins flipped by the mechanism. In contrast, in a truthful-in-expectation mechanism, revealing truthfully her values only maximizes the agent's utility in expectation over the random coins tossed by the mechanism. Therefore, a risk-averse agent modeled by a non-linear utility function may not bid truthfully in a truthful-in-expectation mechanism designed for risk-neutral agents, whereas in a universally truthful mechanism an agent behaves the same regardless of her risk attitude. For this reason, the above-mentioned perspective sees universally truthful mechanisms superior because the performance guarantees shown for risk-neutral agents seem to apply to risk-averse agents as well.

This perspective is incorrect because the optimal performance possible by a mechanism is different for risk-neutral and risk-averse agents. In some cases, a mechanism may exploit the risk attitude of the agents to achieve objectives better than the optimal possible for risk-neutral agents; in other cases, the objective itself relies on the utility functions (e.g. social welfare maximization), and therefore the same outcome has a different objective value. In all these situations, the performance guarantee of universally truthful mechanisms measured by the risk-neutral optimality loses its meaning. We have already discussed above two examples for capacitated agents that illustrate this point: for welfare maximization the second-price auction is not optimal, for revenue maximization the risk-neutral revenue-optimal auction can be far from optimal.

The conclusion of the discussion above is that the universally truthful mechanisms from the literature are not generally good when agents are risk averse; therefore, the solution concept of universal truthfulness buys no additional guarantees over truthfulness in expectation. Nonetheless, our results suggest that it may be possible to develop a general theory for prior-independent mechanisms for risk-averse agents. By necessity, though, this theory will look different from the existing theory of algorithmic mechanism design.

Summary of Results. Our main theorem is that the first-price auction is a prior-independent 5-approximation for revenue for two or more agents with i.i.d. values and risk-averse preferences (given by a common capacity). The technical results that enable this theorem are as follows:

- The optimal auction for agents with capacitated utilities is a two-priced mechanism where a winning agent either pays her full value or her value less her capacity.
- The expected revenue of an agent with capacitated utility and regular value distribution can be bounded in terms of an expected (risk-averse) virtual surplus, where the (risk-averse) virtual value is twice the risk-neutral virtual value plus the value minus capacity (if positive).
- Either the mechanism that optimizes value minus capacity (and charges the Clarke payments or value minus capacity, whichever is higher) or the risk-neutral revenue optimal mechanism is a 3-approximation to the revenue optimal auction for capacitated utilities.
- We characterize the Bayes-Nash equilibria of auctions with capacitated agents where each bidder's payment when served is a deterministic function of her value. An example of this is the first-price auction. The BNE strategies of the capacitated agents can be calculated formulaically from the BNE strategies of risk-neutral agents.

Some of these results extend beyond single-item auctions. In particular, the characterization of equilibrium in the first-price auction holds for position auction environments (i.e., where agents are assigned to positions greedily by bid with decreasing probabilities of service and charged their bid if served). If valuations are symmetric in a position auction, then our prior-independent 5-approximation (Theorem 5.5) holds. Our simple-versus-optimal 3-approximation (Theorem 5.6) holds generally for downward-closed environments, non-identical distributions, and non-identical capacities.

Related Work. The comparative performance of first- and second-price auctions in the presence of risk aversion has been well studied in the Economics literature. From a revenue perspective, first-price auctions are shown to outperform second-price auctions very broadly. Riley and Samuelson [1981] and Holt [1980] show this for symmetric settings where bidders have the same concave utility function. Maskin and Riley [1984] show this for more general preferences.

Matthews [1987] shows that in addition to the revenue dominance, bidders whose risk attitudes exhibit *constant absolute risk aversion (CARA)* are indifferent between first- and second-price auctions, even though they pay more in expectation in the first-price auction. Hu et al. [2010] considers the optimal reserve prices to set in each, and shows that the optimal reserve in the first price auction is less than that in the second price auction. Interestingly, under light conditions on the utility functions, as risk aversion increases, the optimal first-price reserve price decreases.

Matthews [1983] and Maskin and Riley [1984] have considered optimal mechanisms for a single item, with symmetric bidders (i.i.d. values and identical utility function), for CARA and more general preferences. Both approaches apply only when the optimal auction involves a deterministic price upon winning, which is not satisfied in our setting.

Recently, Dughmi and Peres [2012] have shown that by insuring bidders against uncertainty, any truthful-in-expectation mechanism for risk-neutral agents can be converted into a dominant-strategy incentive compatible mechanism for risk-averse buyers with no loss of revenue. However, there is potentially much to gain—mechanisms for risk-averse buyers can achieve unboundedly more welfare and revenue than mechanisms for risk-neutral bidders, as we show in Example 4.3 of Section 4.

2. PRELIMINARIES

Risk-Averse Agents. Consider selling an item to an agent who has a private valuation v drawn from a known distribution F. Denote the outcome by (x, p), where $x \in \{0, 1\}$ indicates whether the agent gets the item, and p is the payment made. The agent obtains a *wealth* of $vx - p$ for such an outcome and the agent's utility is given by a concave utility function $u(\cdot)$ that maps her wealth to utility, i.e., her utility for outcome (x, p) is $u(vx - p)$. Concave utility functions are a standard approach for modeling risk-aversion.[1]

A *capacitated* utility function is $u_C(z) = \min(z, C)$ for a given C which we refer to as the *capacity*. Intuitively, small C relative to value corresponds to severe risk aversion; large C corresponds to mild risk aversion; and $C = \infty$ corresponds to risk neutrality. An agent views an auction as a deterministic rule that maps a random source and the (possibly random) reports of other agents which we summarize by π, and the report b of the agent, to an allocation and payment. We denote these coupled allocation and payment rules as $x^\pi(b)$ and $p^\pi(b)$, respectively. The agent wishes to maximize her expected utility which is given by $\mathbf{E}_\pi[u_C(vx^\pi(b) - p^\pi(b))]$, i.e., she is a von Neumann-Morgenstern utility maximizer.

Incentives. A strategy profile of agents is $\mathbf{s} = (s_1, \ldots, s_n)$ mapping values to reports. Such a strategy profile is in *Bayes-Nash equilibrium* (BNE) if each agent i maximizes her utility

[1] There are other definitions of risk aversion; this one is the least controversial. See Mas-Colell et al. [1995] for a thorough exposition of expected utility theory.

by reporting $s_i(v_i)$. I.e., for all i, v_i, and z:

$$\mathbf{E}_\pi \left[u(v_i x_i^\pi(s_i(v_i)) - p_i^\pi(s_i(v_i)))\right] \geq \mathbf{E}_\pi \left[u(v_i x_i^\pi(z) - p_i^\pi(z))\right]$$

where π denotes the random bits accessed by the mechanism as well as the random inputs $s_j(v_j)$ for $j \neq i$ and $v_j \sim F_j$. A mechanism is *Bayesian incentive compatible* (BIC) if truthtelling is a Bayes-Nash equilibrium: for all i, v_i, and z

$$\mathbf{E}_\pi \left[u(v_i x_i^\pi(v_i) - p_i^\pi(v_i))\right] \geq \mathbf{E}_\pi \left[u(v_i x_i^\pi(z) - p_i^\pi(z))\right] \tag{IC}$$

where π denotes the random bits accessed by the mechanism as well as the random inputs $v_j \sim F_j$ for $j \neq i$.

We will consider only mechanisms where losers have no payments, and winners pay at most their bids. These constraints imply ex post *individual rationality* (IR). Formulaically, for all i, v_i, and π, $p_i^\pi(v_i) \leq v_i$ when $x_i^\pi(v_i) = 1$ and $p_i^\pi(v_i) = 0$ when $x_i^\pi(v_i) = 0$.

Auctions and Objectives. The revenue of an auction $\mathcal{M}$ is the total payment of all agents; its expected revenue for implicit distribution F and Bayes-Nash equilibrium is denoted $\mathrm{Rev}(\mathcal{M}) = \mathbf{E}_{\pi,\mathbf{v}}[\sum_i p_i^\pi(v_i)]$. The welfare of an auction $\mathcal{M}$ is the total utility of all participants including the auctioneer; its expected welfare is denoted $\mathrm{Welfare}(\mathcal{M}) = \mathrm{Rev}(\mathcal{M}) + \mathbf{E}_{\pi,\mathbf{v}}[\sum_i u(v_i x_i^\pi(v_i) - p_i^\pi(v_i))]$.

Some examples of auctions are: the *first-price auction* (FPA) serves the agent with the highest bid and charges her her bid; the *second-price auction* (SPA) serves the agent with the highest bid and charges her the second-highest bid. The second price auction is incentive compatible regardless of agents' risk attitudes. The *capacitated second-price auction* (CSP) serves the agent with the highest bid and charges her the maximum of her value less her capacity and the second highest bid. The second-price auction for capacitated agents is incentive compatible for capacitated agents because, relative to the second-price auction, the utility an agent receives for truthtelling is unaffected and the utility she receives for any misreport is only (weakly) lower.

Two-Priced Auctions. The following class of auctions will be relevant for agents with capacitated utility functions.

Definition 2.1. A mechanism $\mathcal{M}$ is *two-priced* if, whenever $\mathcal{M}$ serves an agent with capacity C and value v, the agent's payment is either v or $v - C$; and otherwise (when not served) her payment is zero. Denote by $x_{\mathrm{val}}(v)$ and $x_C(v)$ probability of paying v and $v - C$, respectively.

Note that from an agent's perspective the outcome of a two-priced mechanism is fully described by a x_C and x_{val}.

Auction Theory for Risk-Neutral Agents. For risk-neutral agents, i.e., with $u(\cdot)$ equal to the identity function, only the probability of winning and expected payment are relevant. The *interim allocation rule* and *interim payment rule* are given by the expectation of x^π and p^π over π and denoted as $x(b) = \mathbf{E}_\pi[x^\pi(b)]$ and $p(b) = \mathbf{E}_\pi[p^\pi(b)]$, respectively (recall that π encodes the randomization of the mechanism and the reports of other agents).

For risk-neutral agents, Myerson [1981] characterized interim allocation and payment rules that arise in BNE and solved for the revenue optimal auction. These results are summarized in the following theorem.

Theorem 2.2 (Myerson 1981). *For risk-neutral bidders with valuations drawn independently and identically from F,*

(1) *(monotonicity) The allocation rule $x(v)$ for each agent is monotone non-decreasing in v.*
(2) *(payment identity) The payment rule satisfies $p(v) = v x(v) - \int_0^v x(z)\mathrm{d}z$.*

(3) *(virtual value) The ex ante expected payment of an agent is* $\mathbf{E}_v[p(v)] = \mathbf{E}_v[\varphi(v)x(v)]$ *where* $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ *is the* virtual value *for value* v*.*

(4) *(optimality) When the distribution* F *is* regular*, i.e.,* $\varphi(v)$ *is monotone, the second-price auction with reserve* $\varphi^{-1}(0)$ *is revenue-optimal.*

The payment identity in part 2 implies the *revenue equivalence* between any two auctions with the same BNE allocation rule.

A well-known result by Bulow and Klemperer shows that, in part 4 of Theorem 2.2, instead of having a reserve price to make the second-price auction optimal, one may as well add in another identical bidder to get at least as much revenue.

THEOREM 2.3 (BULOW AND KLEMPERER 1996). *For risk-neutral bidders with valuations drawn i.i.d. from a regular distribution, the revenue from the second-price auction with* $n+1$ *bidders is at least that of the optimal auction for* n *bidders.*

3. THE OPTIMAL AUCTIONS

In this section we study the form of optimal mechanisms for capacitated agents. In Section 3.1, we show that it is without loss of generality to consider two-priced auctions, and in Section 3.2 we characterize the incentive constraints of two-priced auctions. In Section 3.3 we use this characterization to show that the optimal auction (in discrete type spaces) can be computed in polynomial time in the number of types.

3.1. Two-priced Auctions Are Optimal

Recall a two-priced auction is one where when any agent is served she is either charged her value or her value minus her capacity. We show below that restricting our attention to two-priced auctions is without loss for the objective of revenue.

THEOREM 3.1. *For any BIC auction on capacitated agents (with heterogeneous capacities) there is a two-priced auction with no lower revenue.*

PROOF. We prove this theorem in two steps. In the first step we show, quite simply, that if an agent with a particular value received more wealth than C then we can truncate her wealth to C (by charging her more). With her given value she is indifferent to this change, and for all other values this change makes misreporting this value (weakly) less desirable. Therefore, such a change would not induce misreporting and only (weakly) increases revenue. This first step gives a mechanism wherein every agent's wealth is in the linear part of her utility function. The second step is to show that we can transform the distribution of wealth into a two point distribution. Whenever an agent with value v is offered a price that results in a wealth $w \in [0, C]$, we instead offer her a price of $v - C$ with probability w/C, and a price of v with the remaining probability. Both the expected revenue and the utility of a truthful bidder is unchanged. The expected utility of other types to misreport v, however, weakly decreases by the concavity of u_C, because mixing over endpoints of an interval on a concave function gives less value than mixing over internal points with the same expectation. □

3.2. Characterization of Two-Priced Auctions

In this section we characterize the incentive constraints of two-priced auctions. We focus on the induced two-priced mechanism for a single agent given the randomization π of other agent values and the mechanism. The interim two-priced allocation rule of this agent is denoted by $x(v) = x_{\text{val}}(v) + x_C(v)$.

Lemma 3.2. *A mechanism with two-price allocation rule $x = x_{\text{val}} + x_C$ is BIC if and only if for all v and v^+ such that $v < v^+ \le v + C$,*

$$\frac{x_{\text{val}}(v)}{C} \le \frac{x_C(v^+) - x_C(v)}{v^+ - v} \le \frac{x(v^+)}{C}. \tag{1}$$

Equation (1) can be equivalently written as the following two linear constraints on x_C, for all $v^- \le v \le v^+ \in [v - C, v + C]$:

$$x_C(v^+) \ge x_C(v) + \frac{v^+ - v}{C} \cdot x_{\text{val}}(v), \tag{2}$$

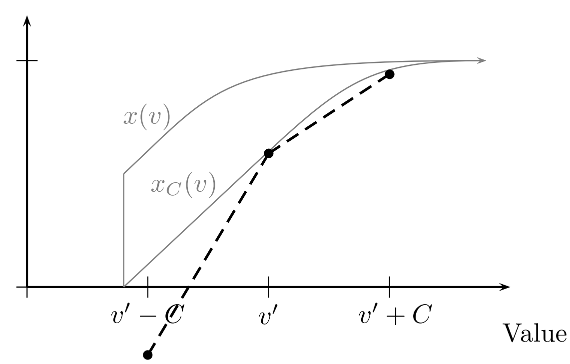
$$x_C(v^-) \ge x_C(v) - \frac{v - v^-}{C} \cdot x(v). \tag{3}$$

Equations (2) and (3) are illustrated in Figure 2. For a fixed v, (2) with $v^+ = v + C$ yields a lower bounding line segment from $(v, x_C(v))$ to $(v + C, x_C(v) + x_{\text{val}}(v))$, and (3) with $v^- = v - C$ gives a lower bounding line segment from $(v, x_C(v))$ to $(v - C, x_C(v) - x(v))$. Note that (2) implies that x_C is monotone.

In the special case when x_C is differentiable, by taking v^+ approaching v in (1), we have $\frac{x_{\text{val}}(v)}{C} \le x'_C(v) \le \frac{x(v)}{C}$ for all v. In general, we have the following condition in the integral form (see the appendix of the full version of the paper for a proof).

Corollary 3.3. *The allocation rule $x = x_{\text{val}} + x_C$ of a BIC two-priced mechanism for all $v < v^+$ satisfies:*

$$\int_v^{v^+} \frac{x_{\text{val}}(z)}{C}\, \mathrm{d}z \le x_C(v^+) - x_C(v) \le \int_v^{v^+} \frac{x(z)}{C}\, \mathrm{d}z. \tag{4}$$



2: Fixing $x(v') = x_{\text{val}}(v') + x_C(v')$, the dashed line between points $(v' - C, x_C(v') - x(v'))$, $(v', x_C(v'))$, and $(v' + C, x_C(v') + x_{\text{val}}(v'))$ (denoted by "•") depicts the lower bounds from (2) and (3) on x_C for values in $[v' - C, v' + C]$.

Importantly, the equilibrium characterization of two-priced mechanisms does not imply monotonicity of the allocation rule x. This is in contrast with mechanisms for risk-neutral agents, where incentive compatibility requires a monotone allocation rule (Theorem 2.2, part 1). This non-monotonicity is exhibited in the following example.

Example 3.4. *There is a single-agent two-priced mechanism with a non-monotone allocation rule.* Our agent has two possible values $v = 3$ and $v = 4$, and capacity C of 2. We give a two price mechanism. Recall that $x_C(v)$ is the probability with which the mechanism sells the item and charges $v - C$; $x_{\text{val}}(v)$ is the probability with which the mechanism sells the item and charges v; and $x(v) = x_C(v) + x_{\text{val}}(v)$. The mechanism and its outcome are summarized in the following table.

v	x	x_C	x_{val}	utility from truthful reporting	utility from misreporting
3	5/6	1/2	1/3	1	2/3
4	2/3	2/3	0	4/3	4/3

3.3. Optimal Auction Computation

Solving for the optimal mechanism is computationally tractable for any discrete (explicitly given) type space T. Given a discrete valuation distribution on support T, one can use $2\,|T|$ variables to represent the allocation rule of any two-priced mechanism, and the expected revenue is a linear sum of these variables. Lemma 3.2 shows that one can use $O(|T|^2)$ linear constraints to express all BIC allocations, and hence the revenue optimization for a single bidder can be solved by a $O(|T|^2)$-sized linear program. Furthermore, using techniques developed by Cai et al. [2012] and Alaei et al. [2012], in particular the "token-passing" characterization of single-item auctions by Alaei et al. [2012], we obtain:

THEOREM 3.5. *For n bidders with independent valuations with type spaces $T_1, \cdots, T_n$ and capacities $C_1, \cdots, C_n$, one can solve for the optimal single-item auction with a linear program of size $O\Big((\sum_i |T_i|)^2\Big)$.*

4. AN UPPER BOUND ON TWO-PRICED EXPECTED PAYMENT

In this section we will prove an upper bound on the expected payment from any capacitated agent in a two-priced mechanism. This upper bound is analogous in purpose to the identity between expected risk-neutral payments and expected virtual surplus of Myerson [1981] from which optimal auctions for risk-neutral agents are derived. We use this bound in Section 5.2 and Section 5.3 to derive approximately optimal mechanisms.

As before, we focus on the induced two-priced mechanism for a single agent given the randomization π of other agent values and the mechanism. The expected payment of a bidder of value v under allocation rule $x(v) = x_C(v) + x_{\text{val}}(v)$ is $p(v) = v \cdot x_{\text{val}}(v) + (v - C) \cdot x_C(v) = v \cdot x(v) - C \cdot x_C(v)$.

Recall from Theorem 2.2 that the (risk-neutral) virtual value for an agent with value drawn from distribution F is $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and that the expected risk-neutral payment for allocation rule $x(\cdot)$ is $\mathbf{E}_v[\varphi(v)x(v)]$. Denote $\max(0, \varphi(v))$ by $\varphi^+(v)$ and $\max(v - C, 0)$ by $(v - C)^+$.

THEOREM 4.1. *For any agent with value $v \sim F$, capacity C, and two-priced allocation rule $x(v) = x_{\text{val}}(v) + x_C(v)$,*

$$\begin{aligned}\mathbf{E}_v\left[p(v)\right] \le \mathbf{E}_v\Big[\varphi^+(v) \cdot x(v)\Big] + \mathbf{E}_v\Big[\varphi^+(v) \cdot x_C(v)\Big] \\ + \mathbf{E}_v\Big[(v - C)^+ \cdot x_C(v)\Big].\end{aligned}$$

COROLLARY 4.2. *When bidders have regular distributions and a common capacity, either the risk-neutral optimal auction or the capacitated second price auction (whichever has higher revenue) gives a 3-approximation to the optimal revenue for capacitated agents.*

PROOF. For each of the three parts of the revenue upper bound of Theorem 4.1, there is a simple auction that optimizes the expectation of the part across all agents. For the first two parts, the allocation rules across agents (both for $x(\cdot)$ and $x_C(\cdot)$) are feasible. When the distributions of agent values are regular (i.e., the virtual value functions are monotone), the risk-neutral revenue-optimal auction optimizes virtual surplus across all feasible allocations (i.e., expected virtual value of the agent served); therefore, its expected revenue upper bounds the first and second parts of the bound in Theorem 4.1. The revenue of the third part is again the expectation of a monotone function (in this case $(v - C)^+$) times the service probability. The auction that serves the agent with the highest (positive) "value minus capacity" (and charges the winner the maximum of her "minimum winning bid," i.e., the second-price payment rule, and her "value minus capacity") optimizes such an expression over all feasible allocations; therefore, its revenue upper bounds this third part of the bound in Theorem 4.1. When capacities are identical, this auction is the capacitated second price auction. □

Before proving Theorem 4.1, we give two examples. The first shows that the gap between the revenue of the capacitated second-price auction and the risk-neutral revenue-optimal auction (i.e., the two auctions from Corollary 4.2) can be arbitrarily large. This means that there is no hope that an auction for risk-neutral agents always obtains good revenue for risk-averse agents. The second example shows that even when all values are bounded from above by the capacity (and therefore, capacities are never binding in a risk-neutral auction) an auction for risk-averse agents can still take advantage of risk aversion to generate higher revenue. Consequently, the fact that we have two risk-neutral revenue terms in the bound of Theorem 4.1 is necessary (as the "value minus capacity" term is zero in this case).

Example 4.3. *Auctions with capacitated players can achieve unboundedly more revenue than with risk-neutral players.* The *equal revenue* distribution on interval $[1, h]$ has distribution function $F(z) = 1 - 1/z$ (with a point mass at h). The distribution gets its name because such an agent would accept any offer price of p with probability $1/p$ and generate an expected revenue of one. With one such agent the optimal risk-neutral revenue is one. Of course, an agent with capacity $C = 1$ would happily pay her value minus her capacity to win all the time (i.e., $x(v) = x_C(v) = 1$). The revenue of this auction is $\mathbf{E}[v] - 1 = \ln h$. For large h, this is unboundedly larger than the revenue we can obtain from a risk-neutral agent with the same distribution.

Example 4.4. *The revenue from a two-priced mechanism can be better than the optimal risk-neutral revenue even when all values are no more than the capacity.* Consider selling to an agent with capacity of $C = 1000$ and value drawn from the equal revenue distribution from Example Example 4.3 with $h = 1000$.

The following two-priced rule is BIC and generates revenue of approximately 1.55 when selling to such a bidder. Let $x_C(v) = \frac{0.6}{1000}(v - 1)$, $x(v) = \min(x_C(v) + 0.6, 1)$, and $x_{\text{val}}(v) = x(v) - x_C(v)$ (shown in Figure 3). Recall that the expected payment from an agent with value v can be written as $vx(v) - Cx_C(v)$; for small values, this will be approximately 0.6; for large values this will increase to 400. The expected revenue is $\int_1^{1000} \big(z \cdot x(z) - 1000\, x_C(z)\big) f(z)\mathrm{d}z + \frac{1}{1000}(1000 \cdot x_{\text{val}}(1000)) \approx 1.15 + 0.4 \approx 1.55$, an improvement over the optimal risk-neutral revenue of 1.

In the remainder of this section we instantiate the following outline for the proof of Theorem 4.1. First, we transform any given two-priced allocation rule $x = x_{\text{val}} + x_C$ into a new two-priced rule $\bar{x}(v) = \bar{x}_C(v) + \bar{x}_{\text{val}}(v)$ (for which the expected payment is $\bar{p}(v) = v\bar{x}(v) - C\bar{x}_C(v)$). While this transformation may violate some incentive constraints (from Lemma 3.2), it enforces convexity of $\bar{x}_C(v)$ on $v \in [0, C]$ and (weakly) improves revenue. Second, we derive a simple upper bound on the payment rule $\bar{p}(\cdot)$. Finally, we use the enforced convexity property of $\bar{x}_C(\cdot)$ and the revenue upper bound to partition the expected payment $\mathbf{E}_v[\bar{p}(v)]$ by the three terms that can each be attained by simple mechanisms.

4.1. Two-Priced Allocation Construction

We now construct a two-priced allocation rule $\bar{x} = \bar{x}_{\mathrm{val}} + \bar{x}_C$ from $x = x_{\mathrm{val}} + x_C$ for which (a) revenue is improved, i.e., $\bar{p}(v) \geq p(v)$, and (b) the probability the agent pays her value minus capacity, $\bar{x}_C(v)$, is convex for $v \in [0, C]$. In fact, given x_{val}, $\bar{x}_C$ is the smallest function for which IC constraint (2) holds; and in the special case when x_{val} is monotone, the left-hand side of (4) is tight for $\bar{x}_C$ on $[0, C]$. Other incentive constraints may be violated by $\bar{x}$, but we use it only as an upper bound for revenue.

Definition 4.5 ($\bar{x}$). We define $\bar{x} = \bar{x}_C + \bar{x}_{\mathrm{val}}$ as follows:

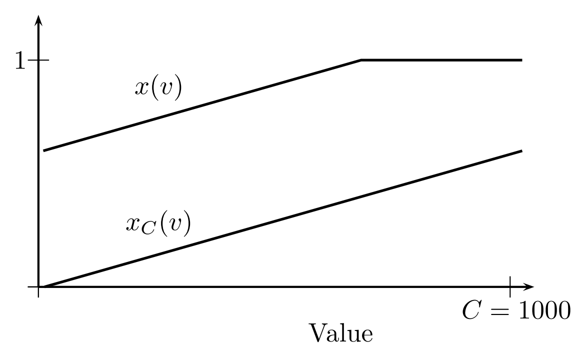
(1) $\bar{x}_{\mathrm{val}}(v) = x_{\mathrm{val}}(v)$;
(2) Let $r(v)$ be $\frac{1}{C}\sup_{z \leq v} x_{\mathrm{val}}(z)$, and let

$$\bar{x}_C(v) = \begin{cases} \int_0^v r(y)\,\mathrm{d}y, & v \in [0, C]; \\ x_C(v), & v > C. \end{cases} \tag{5}$$

LEMMA 4.6 (PROPERTIES OF $\bar{x}$).

(1) *On $v \in [0, C]$, $\bar{x}_C(\cdot)$ is a convex, monotone increasing function.*
(2) *On all v, $\bar{x}_C(v) \leq x_C(v)$.*
(3) *The incentive constraint from the left-hand side of (4) holds for $\bar{x}_C$: $\frac{1}{C}\int_v^{v^+} \bar{x}_{\mathrm{val}}(z)\,\mathrm{d}z \leq \bar{x}_C(v^+) - \bar{x}_C(v)$ for all $v < v^+$.*
(4) *On all v, $\bar{x}_C(v) \leq x_C(v)$, $\bar{x}(v) \leq x(v)$, and $\bar{p}(v) \geq p(v)$.*

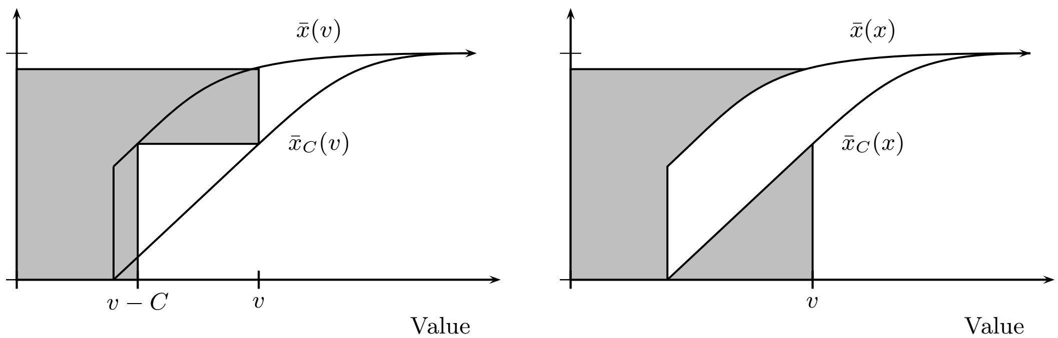
The proof of part 2 is technical, and we give a sketch here. Recall that, for each v, the IC constraint (2) gives a linear constraint lower bounding $x_C(v^+)$ for every $v^+ > v$. If one decreases $x_C(v)$, the lower bound it imposes on $x_C(v^+)$ is simply "pulled down" and is less binding. The definition of $\bar{x}_C$ simply lands $\bar{x}_C(v)$ on the most binding lower bound, and therefore not only makes $\bar{x}_C(v)$ at most $x_C(v)$, but also lowers the linear constraint that v imposes on larger values. If the number of values is countable or if x_{val} is piecewise constant, the lemma is easy to see by induction. A full proof for the general case of part 2, along with



3: With $C = 1000$ and values drawn from the equal revenue distribution on $[1, 1000]$, this two-priced mechanism is BIC and achieves 1.55 times the revenue of the optimal risk-neutral mechanism.

the proofs of the other more direct parts of Lemma 4.6, is given in the appendix of the full version of the paper.

4.2. Payment Upper Bound



(a) Shaded region is the expected payment from an agent of value v.

(b) Shaded region upper bounds expected payment from an agent with value v, shown in Lemma 4.7.

4:

Recall that $\bar{p}(v)$ is the expected payment corresponding with two-priced allocation rule $\bar{x}(v)$. We now give an upper bound on $\bar{p}(v)$.

Lemma 4.7. *The payment $\bar{p}(v)$ for v and two-priced rule $\bar{x}(v)$ satisfies*

$$\bar{p}(v) \leq v\bar{x}(v) - \int_0^v \bar{x}(z)\,\mathrm{d}z + \int_0^v \bar{x}_C(z)\,\mathrm{d}z. \tag{6}$$

Proof. View a two-priced mechanism $\bar{x} = \bar{x}_{\text{val}} + \bar{x}_C$ as charging v with probability $\bar{x}(v)$ and giving a rebate of C with probability $\bar{x}_C(v)$. We bound this rebate as follows (which proves the lemma):

$$\begin{aligned} C \cdot \bar{x}_C(v) &\geq C \cdot \bar{x}_C(0) + \int_0^v \bar{x}_{\text{val}}(z)\,\mathrm{d}z \\ &\geq \int_0^v \bar{x}(z)\,\mathrm{d}z - \int_0^v \bar{x}_C(z)\,\mathrm{d}z. \end{aligned}$$

The first inequality is from part 3 of Lemma 4.6. The second inequality is from the definition of $\bar{x}_C(0) = 0$ in (5) and $\bar{x}_{\text{val}}(v) = \bar{x}(v) - \bar{x}_C(v)$. See Figure 4 for an illustration. □

4.3. Three-part Payment Decomposition

Below, we bound $\bar{p}(\cdot)$ (and hence $p(\cdot)$) in terms of the expected payment of three natural mechanisms. As seen geometrically in Figure 5, the bound given in Lemma 4.7 can be broken into two parts: the area above $\bar{x}(\cdot)$, and the area below $\bar{x}_C(\cdot)$. We refer to the former as $\bar{p}^{\mathrm{I}}(\cdot)$; we further split the latter quantity into two parts: $\bar{p}^{\mathrm{II}}(\cdot)$, the area corresponding to

$v \in [0, C]$, and $\bar{p}^{\mathrm{III}}(\cdot)$, that corresponding to $v \in [C, v]$. We define these quantities formally below:

$$\bar{p}^{\mathrm{I}}(v) = \bar{x}(v)v - \int_0^v \bar{x}(z)\,\mathrm{d}z, \tag{7}$$

$$\bar{p}^{\mathrm{II}}(v) = \int_0^{\min\{v,C\}} \bar{x}_C(z)\,\mathrm{d}z \tag{8}$$

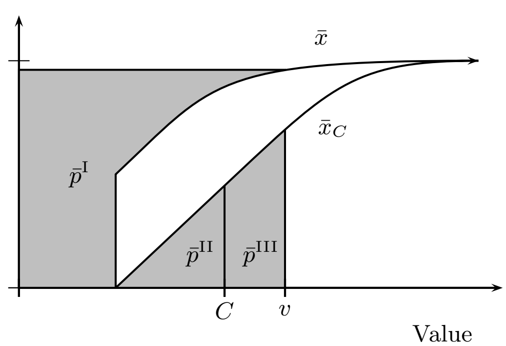
$$\bar{p}^{\mathrm{III}}(v) = \begin{cases} 0, & v \leq C; \\ \int_C^v \bar{x}_C(z)\,\mathrm{d}z, & v > C. \end{cases} \tag{9}$$

Proof of Theorem 4.1. We now bound the revenue from each of the three parts of the payment decomposition. These bounds, combined with part 4 of Lemma 4.6 and Lemma 4.7, immediately give Theorem 4.1.

Part 1. $\mathbf{E}_v[\bar{p}^{\mathrm{I}}(v)] = \mathbf{E}_v[\varphi(v) \cdot \bar{x}(v)] \leq \mathbf{E}_v[\varphi^+(v) \cdot x(v)]$.
Formulaically, $\bar{p}^{\mathrm{I}}(\cdot)$ corresponds to the risk-neutral payment identity for $\bar{x}(\cdot)$ as specified by part 2 of Theorem 2.2; by part 3 of Theorem 2.2, in expectation over v, this payment is equal to the expected virtual surplus $\mathbf{E}_v[\varphi(v) \cdot \bar{x}(v)]$.[2] The inequality follows as terms $\varphi(v)$ and $\bar{x}(v)$ in this expectation are point-wise upper bounded by $\varphi^+(v) = \max(\varphi(v), 0)$ and $x(v)$, respectively, the latter by part 4 of Lemma 4.6.

Part 2. $\mathbf{E}_v[\bar{p}^{\mathrm{II}}(v)] \leq \mathbf{E}_v[\varphi(v) \cdot \bar{x}_C(v)] \leq \mathbf{E}_v[\varphi^+(v) \cdot x_C(v)]$.
By definition of $\bar{p}^{\mathrm{II}}(\cdot)$ in (8), if the statement holds for $v = C$ it holds for $v > C$; so we argue it only for $v \in [0, C]$. Formulaically, with respect to a risk-neutral agent with allocation rule $\bar{x}_C(\cdot)$, the risk-neutral payment is $v \cdot \bar{x}_C(v) - \int_0^v \bar{x}_C(z)\,\mathrm{d}z$, the surplus is $v \cdot \bar{x}_C(v)$, and the risk-neutral agent's utility (the difference between the surplus and payment) is $\int_0^v \bar{x}_C(z)\,\mathrm{d}z = \bar{p}^{\mathrm{II}}(v)$. Convexity of $\bar{x}_C(\cdot)$, from part 1 of Lemma 4.6, implies that the risk-neutral payment is at least half the surplus, and so is at least the risk-neutral utility. The lemma follows, then, by the same argument as in the previous part.

[2]Note: This equality does not require monotonicity of the allocation rule $\bar{x}(\cdot)$; as long as part 2 of Theorem 2.2 formulaically holds, part 3 follows from integration by parts.



5: Breakdown of the expected payment upper bound in a two-priced auction.

Part 3. $\mathbf{E}_v[\bar{p}^{\mathrm{III}}(v)] \le \mathbf{E}_v[(v-C)^+ \cdot \bar{x}_C(v)] = \mathbf{E}_v[(v-C)^+ \cdot x_C(v)]$.

The statement is trivial for $v \le C$ so assume $v \ge C$. By definition $\bar{x}_C(v) = x_C(v)$ for $v > C$. By (2), $x_C(\cdot)$ is monotone non-decreasing. Hence, for $v > C$, $\bar{p}^{\mathrm{III}}(v) = \int_C^v x_C(z)\,\mathrm{d}z \le \int_C^v x_C(v)\,\mathrm{d}z = (v-C)\cdot x_C(v)$. Plugging in $(v-C)^+ = \max(v-C, 0)$ and taking expectation over v, we obtain the bound.

□

5. APPROXIMATION MECHANISMS AND A PAYMENT IDENTITY

In this section we first give a payment identity for Bayes-Nash equilibria in mechanisms that charge agents a deterministic amount upon winning (and zero upon losing). Such one-priced payment schemes are not optimal for capacitated agents; however, we will show that they are approximately optimal. When agents are symmetric (with identical distribution and capacity) we use this payment identity to prove that the first-price auction is approximately optimal. When agents are asymmetric we give a simple direct-revelation one-priced mechanism that is BIC and approximately optimal.

5.1. A One-price Payment Identity

For risk-neutral agents, the Bayes-Nash equilibrium conditions entail a payment identity: given an interim allocation rule, the payment rule is fixed (Theorem 2.2, part 2). For risk-averse agents there is no such payment identity: there are mechanisms with the identical BNE allocation rules but distinct BNE payment rules. We restrict attention to auctions wherein an agent's payment is a deterministic function of her value (if she wins) and zero if she loses. We call these *one-priced* mechanisms; for these mechanisms there is a (partial) payment identity.

We consider a single agent and the induced allocation rule she faces from a Bayesian incentive compatible auction (or, by the revelation principle, any BNE of any mechanism). This allocation rule internalizes randomization in the environment and the auction, and specifies the agents' probability of winning, $x(v)$, as a function of her value. Given allocation rule $x(v)$, the risk-neutral expected payment is $p^{\mathrm{RN}}(v) = v\cdot x(v) - \int_0^v x(z)\,\mathrm{d}z$ (Theorem 2.2, part 2). Given an allocation rule $x(v)$, a one-priced mechanism with payment rule $p(v)$ would charge the agent $p(v)/x(v)$ upon winning and zero otherwise (for an expected payment of $p(v)$). Define $p^{\mathrm{VC}}(v) = (v-C)\cdot x(v)$ which, intuitively, gives a lower bound on a capacitated agent's willingness to trade-off decreased probability of winning for a cheaper price.

THEOREM 5.1. *An allocation rule x and payment rule p are the BNE of a one-priced mechanism if and only if (a) x is monotone non-decreasing and (b) if $p(v) \ge p^{\mathrm{VC}}(v)$ for all v then $p = p^C$ is defined as*

$$p^C(0) = 0, \tag{10}$$

$$p^C(v) = \max\left(p^{\mathrm{VC}}(v),\ \sup_{v^- < v}\left\{p^C(v^-) + (p^{\mathrm{RN}}(v) - p^{\mathrm{RN}}(v^-))\right\}\right). \tag{11}$$

Moreover, if x is strictly increasing then $p(v) \ge p^{\mathrm{VC}}(v)$ for all v and $p = p^C$ is the unique equilibrium payment rule.

The payment rule should be thought of in terms of two "regimes": when $p^C = p^{\mathrm{VC}}$, and when $p^C > p^{\mathrm{VC}}$, corresponding to the first and second terms in the max argument of (11) respectively. In the latter regime, (11) necessitates that $\frac{\mathrm{d}}{\mathrm{d}v}p^C(v) = \frac{\mathrm{d}}{\mathrm{d}v}p^{\mathrm{RN}}(v)$; for nearby such points v and $v+\epsilon$, the v^- involved in the supremum will be the same, and thus $p^C(v+\epsilon) - p^C(v) = p^{\mathrm{RN}}(v+\epsilon) - p^{\mathrm{RN}}(v)$.

The proof is relegated to the appendix of the full version of the paper. The main intuition for this characterization is that risk-neutral payments are "memoryless" in the following sense. Suppose we fix $p^{\mathrm{RN}}(v)$ for a v and ignore the incentive of an agent with value $v^+ > v$ to prefer reporting $v^- < v$, then the risk-neutral payment for all $v^+ > v$ is $p^{\mathrm{RN}}(v^+) = p(v) + \int_v^{v^+} (x(v^+) - x(z))\,\mathrm{d}z$. This memorylessness is simply the manifestation of the fact that the risk-neutral payment identity imposes local constraints on the derivatives of the payment, i.e., $\frac{\mathrm{d}}{\mathrm{d}v} p^{\mathrm{RN}}(v) = v \cdot \frac{\mathrm{d}}{\mathrm{d}v} x(v)$.

There is a simple algorithm for constructing the risk-averse payment rule p^C from the risk-neutral payment rule p^{RN} (for the same allocation rule x).

(0) For $v < C$, $p^C(v) = p^{\mathrm{RN}}(v)$.
(1) The $p^C(v) = p^{\mathrm{RN}}(v)$ identity continues until the value v' where $p^C(v') = p^{\mathrm{VC}}(v')$, and $p^C(v)$ switches to follow $p^{\mathrm{VC}}(v)$.
(2) When v increases to the value v'' where $\frac{\mathrm{d}}{\mathrm{d}v} p^{\mathrm{RN}}(v'') = \frac{\mathrm{d}p^{\mathrm{VC}}}{\mathrm{d}v}(v'')$ then $p^C(v)$ switches to follow $p^{\mathrm{RN}}(v)$ shifted up by the difference $p^{\mathrm{VC}}(v'') - p^{\mathrm{RN}}(v'')$ (i.e., its derivative $\frac{\mathrm{d}}{\mathrm{d}v} p^C(v)$ follows $\frac{\mathrm{d}}{\mathrm{d}v} p^{\mathrm{RN}}(v)$).
(3) Repeat this process from Step 1.

Lemma 5.2. *The one-priced BIC allocation rule x and payment rule p^C satisfy the following*

(*1*) *For all v, $p^C(v) \geq \max(p^{\mathrm{RN}}(v), p^{\mathrm{VC}}(v))$.*
(*2*) *Both $p^C(v)$ and $p^C(v)/x(v)$ are monotone non-decreasing.*

The proof of part 2 is contained in the the appendix of the full version of the paper, and part 1 follows directly from equations (10) and (11).

5.2. Approximate Optimality of First-price Auction

We show herein that for agents with a common capacity and values drawn i.i.d. from a continuous, regular distribution F with strictly positive density the first-price auction is approximately optimal.

It is easy to solve for a symmetric equilibrium in the first-price auction with identical agents. First, guess that in BNE the agent with the highest value wins. When the agents are i.i.d. draws from distribution F, the implied allocation rule is $x(v) = F^{n-1}(v)$. Theorem 5.1 then gives the necessary equilibrium payment rule $p^C(v)$ from which the bid function $b^C(v) = p^C(v)/x(v)$ can be calculated. We verify that the initial guess is correct as Lemma 5.2 implies that the bid function is symmetric and monotone. There is no other symmetric equilibrium.[3] Moreover, **?** show that there are no asymmetric equilibria of the first-price auction for a class of environments including ours.

Proposition 5.3. *The first-price auction for identical (capacity and value distribution) agents has a unique BNE wherein the highest valued agent wins.*

The expected revenue at this equilibrium is $n\,\mathbf{E}_v[p^C(v)]$. Lemma 5.2 implies that p^C is at least p^{RN} and p^{VC}.

Corollary 5.4. *The expected revenue of the first-price auction for identical (capacity and value distribution) agents is at least that of the capacitated second-price auction and at least that of the second-price auction.*

[3]Any other symmetric equilibrum must have an allocation rule that is increasing but not always strictly so. For this to occur the bid function must not be strictly increasing implying a point mass in the distribution of bids. Of course, a point mass in a symmetric equilibrium bid function implies that a tie is not a measure zero event. Any agent has a best response to such an equilibrium of bidding just higher than this pointmass so at essentially the same payment, she always "wins" the tie.

In conjunction with Theorem 2.3 by Bulow and Klemperer [1996], this now allows us to use the capacitated first-price auction to approximate each of the three terms in the revenue bound of Theorem 4.1, giving us our main result:

THEOREM 5.5. *For $n \geq 2$ agents with common capacity and values drawn i.i.d. from a regular distribution, the revenue in the first price auction (FPA) in the Bayes-Nash equilibrium is a 5-approximation to the optimal revenue.*

PROOF. An immediate consequence of Theorem 2.3 is that for $n \geq 2$ risk-neutral, regular, i.i.d. bidders, the second-price auction extracts a revenue that is at least half the optimal revenue; hence, by Corollary 4.2, the optimal revenue for capacitated bidders by any BIC mechanism is at most four times the second-price revenue plus the capacitated second-price revenue. Since the first-price auction revenue in BNE for capacitated agents is at least the capacitated second-price revenue and the second-price revenue, the first-price revenue is a 5-approximation to the optimal revenue.[4] □

5.3. Approximate Optimality of One-Price Auctions in Asymmetric Settings

We now consider the case of asymmetric value distributions and capacities. In such settings the highest-bid-wins first-price auction does not have a symmetric equilibria and arguing revenue bounds for it is difficult. Nonetheless, we can give asymmetric one-priced revelation mechanisms that are BIC and approximately optimal. With respect to Example 4.4 in Section 4 which shows that the option to charge two possible prices from a given type may be necessary for optimal revenue extraction, this result shows that charging two prices over charging one price does not confer a significant advantage.

THEOREM 5.6. *For n (non-identical) agents with capacities $C_1, \ldots, C_n$, and regular value distributions $F_1, \ldots, F_n$, there is a one-priced BIC mechanism whose revenue is at least one third of the optimal (two-priced) revenue.*

PROOF. Recall from Theorem 4.1 that either the risk-neutral optimal revenue or $\mathbf{E}_{v_1,\ldots,v_n}[\max\{(v_i - C_i)_+\}]$ is at least one third of the optimal revenue. We apply Theorem 5.1 to two monotone allocation rules for each player:

(1) the interim allocation rule of the risk-neutral optimal auction, and
(2) the interim allocation rule specified by: serve agent i that maximizes $v_i - C_i$, if positive; otherwise, serve nobody.

As both allocations are monotone, we apply Theorem 5.1 to obtain two single-priced BIC mechanisms. By Lemma 5.2, the expected revenue of the first mechanism is at least the risk neutral optimal revenue, and the expected revenue of the second mechanism is at least $\mathbf{E}_{v_1,\ldots,v_n}[\max\{(v_i - C_i)_+\}]$. The theorem immediately follows for the auction with the higher expected revenue. □

Although Theorem 5.6 is stated as an existential result, the two one-priced mechanisms in the proof can be described analytically using the algorithm following Theorem 5.1 for calculating the capacitated BIC payment rule. The interim allocation rules are straightforward (the first: $x_i(v_i) = \prod_{j \neq i} F_j(\varphi_j^{-1}(\varphi_i(v_i)))$, and the second: $x_i(v_i) = \prod_{j \neq i} F_j(v_i - C_i + C_j)$), and from these we can solve for $p_i^{C_i}(v)$.

6. CONCLUSIONS

For the purpose of keeping the exposition simple, we have applied our analysis only to single-item auctions. Our techniques, however, as they focus on analyzing and bounding

[4]In fact, the Bulow and Klemperer [1996] result shows that the second-price auction is asymptotically optimal; using this, our result can be tightened to a $(3 + \frac{2}{n-1})$-approximation.

revenue of a single agent for a given allocation rule, generalize easily to structurally rich environments. Notice that the main theorems of Sections 3, 4, and the first part of Section 5 do not rely on any assumptions on the feasibility constraint except for downward closure, i.e., that it is feasible to change an allocation by withholding service to an agent who was formerly being served.

For example, our prior-independent 5-approximation result (Theorem 5.5) generalizes to symmetric feasibility constraints such as position auctions. A position auction environment is given by a decreasing sequence of weights $\alpha_1, \ldots, \alpha_n$ and the first-price position auction assigns the agents to these positions greedily by bid. With probability α_i the agent in position i receives an item and is charged her bid; otherwise she is not charged. (These position auctions have been used to model pay-per-click auctions for selling advertisements on search engines where α_i is the probability that an advertiser gets clicked on when her ad is shown in the ith position on the search results page.) For agents with identical capacities and value distributions, the first-price position auction where the bottom half of the agents are always rejected is a 5-approximation to the revenue-optimal position auction (that may potentially match all the agents to slots).

Our one- versus two-price result (Theorem 5.6) generalizes to asymmetric capacities, asymmetric distributions, and asymmetric downward-closed feasibility constraints. A downward-closed feasibility constraint is given by a set system which specifies which subset of agents can be simultaneously served. Downward-closure requires that any subset of a feasible set is feasible. A simple one-priced mechanism is a 3-approximation to the optimal mechanism in such an environment. The mechanism is whichever has higher revenue of the standard (risk neutral) revenue-optimal mechanism (which serves the subset of agents with the highest virtual surplus, i.e., sum of virtual values) and the one-priced revelation mechanism that serves the set of agents S that maximizes $\sum_{i \in S}(v_i - C_i)^+$ subject to feasibility.

A main direction for future work is to relax some of the assumptions of our model. Our approach to optimizing over mechanisms for risk-averse agents relies on (a) the simple model of risk aversion given by capacitated utilities and (b) that losers neither make (i.e., ex post individual rationality) nor receive payments (i.e., no bribes). These restrictions are fundamental for obtaining linear incentive compatibility constraints. Of great interest in future study is relaxation of these assumptions.

There is a relatively well-behaved class of risk attitudes known as *constant absolute risk aversion* where the utility function is parameterized by risk parameter R as $u_R(w) = \frac{1}{R}(1 - e^{-Rw})$. These model the setting in which a bidder's risk aversion is independent of wealth, and hence bidders view a lottery over payments for an item the same no matter their valuations. Matthews [1984] exploits this and derives the optimal auction for such risk attitudes. A first step in extending our results to more interesting risk attitudes would be to consider such risk preferences.

Our analytical (and computational) solution to the optimal auction problem for agents with capacitated utilities requires an *ex post individual rationality* constraint on the mechanism that is standard in algorithmic mechanism design. This constraint requires that an agent who loses the auction cannot be charged. While such a constraint is natural in many settings, it is with loss and, in fact, ill motivated for settings with risk-averse agents. One of the most standard mechanisms for agents with risk-averse preferences is the "insurance mechanism" where an agent who may face some large liability with small probability will prefer to pay a constant insurance fee so that the insurance agency will cover the large liability in the event that it is incurred. This mechanism is not ex post individually rational. Does the first-price auction (which is ex post individual rational) approximate the optimal interim individually rational mechanism?

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