On Budget-Balanced Group-Strategyproof Cost-Sharing Mechanisms

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Abstract. A cost-sharing mechanism defines how to share the cost of a service among serviced customers. It solicits bids from potential customers and selects a subset of customers to serve and a price to charge each of them. The mechanism is group-strategyproof if no subset of customers can gain by lying about their values. There is a rich literature that designs group-strategyproof cost-sharing mechanisms using schemes that satisfy a property called *cross-monotonicity*. Unfortunately, Immorlica et al. showed that for many services, cross-monotonic schemes are provably not *budget-balanced*, i.e., they can recover only a fraction of the cost. While cross-monotonicity is a sufficient condition for designing group-strategyproof mechanisms, it is not necessary. Pountourakis and Vidali recently provided a complete characterization of group-strategyproof mechanisms. We construct a fully budget-balanced cost-sharing mechanism for the edge-cover problem that is not cross-monotonic and we apply their characterization to show that it is group-strategyproof. This improves upon the cross-monotonic approach which can recover only half the cost, and provides a proof-of-concept as to the usefulness of the complete characterization. This raises the question of whether all "natural" problems have budget-balanced group-strategyproof mechanisms. We answer this question in the negative by designing a set-cover instance in which no group-strategyproof mechanism can recover more than a (18/19)-fraction of the cost.

1 Introduction

In *cost-sharing* problems, a service provider faces a set of potential customers, each of which has a private value for the service. The provider must select a subset of customers to serve, and a price to charge each of them. To this end, he defines a mechanism that solicits bids from potential customers and, based on these bids, outputs the serviced subset and corresponding prices. To cover the cost of providing service, he looks for a mechanism that is *budget-balanced*, that is the sum of prices equals the cost of service for every input bid vector.

A central goal in mechanism design is to define mechanisms that are *strategyproof* in that no agent can gain by misreporting his value. This guarantees that the equilibrium bidding strategy of agents is robust and so the mechanism

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behaves as predicted. In cost-sharing problems, there is an inherent cooperative aspect: the cost of service changes drastically depending on which subset is serviced and so groups of agents may have aligned interests. In these problems, it makes sense to ask for an even more robust solution concept, *group-strategyproofness*. In a group-strategyproof mechanism, no group of agents can mutually gain by misreporting their values.

Group-strategyproofness is a very strong requirement. Nonetheless, there is a rich literature defining group-strategyproof mechanisms for various cost-sharing problems [5, 8, 7, 12]. All these papers use the same general technique. They define a *cost-sharing scheme* which, given any subset of customers, defines the price each of them would have to pay if that subset was serviced. They then turn this scheme into a mechanism by applying a procedure of Moulin [10]. The resulting mechanism is group-strategyproof so long as the underlying cost-sharing scheme satisfies a property called *cross-monotonicity*. Intuitively, cross-monotonicity requires that as more agents are serviced, the price to each decreases. If the cross-monotonic cost-sharing scheme is (approximately) budget-balanced on every subset of customers, then the resulting group-strategyproof mechanism is also (approximately) budget-balanced.

Unfortunately, the use of this technique comes at a cost. While submodular cost functions always have fully budget-balanced cross-monotonic cost-sharing schemes [11], and many combinatorial optimization problems have approximately budget-balanced schemes [5, 8, 7, 12], Immorlica et al. [6] showed that cross-monotonicity fundamentally limits achievable budget-balance factors for many combinatorial optimization problems. They also note that, while cross-monotonicity is a sufficient condition for giving rise to group-strategyproof mechanisms, it is not necessary. This left open the question of whether another approach might enable the design of group-strategyproof mechanisms with better budget-balance factors.

In recent work Pountourakis and Vidali [14] provided a complete characterization of group-strategyproof mechanisms. Their characterization is based on cost-sharing schemes that satisfy three technical properties. They then give a procedure that converts any such (approximately) budget-balanced cost-sharing scheme into an (approximately) budget-balanced group-strategyproof mechanism.[1]

In this work, we provide a natural cost-sharing scheme for the edge-cover problem and use the techniques of Pountourakis-Vidali to prove it gives rise to a group-strategyproof mechanism. In the edge-cover problem, the agents are the vertices of a graph, and the cost of a subset is the minimum number of edges that must be selected in order to cover every agent in the subset. The problem models, for example, assigning people to rooms either as a single occupant or with a compatible roommate (as defined by the edges of the graph).

It is shown that the best budget-balance factor of any cross-monotonic cost-sharing scheme for edge-cover is just $1/2$ [6]. Using the complete characterization [14], we design a *fully* budget-balanced group-strategyproof mechanism for

[1] In general, this procedure is not known to be polynomial-time.

edge-cover. Thus our result improves upon *any* group-strategyproof mechanism designed using the standard cross-monotonic technique, thereby demonstrating the significance of the full characterization. Furthermore, the cost-sharing scheme that we define is very intuitive: for a given subset of agents, we compute a lexicographically first maximum matching of that subset, charge each matched agent a price of $1/2$, and charge each remaining agent in the subset a price of 1. This natural scheme is not cross-monotonic. However, using a key lemma regarding alternating paths of certain matchings, we are able to prove that this scheme does satisfy the characterization of Pountourakis and Vidali [14].

We would like to stress out that we wanted to provide a simple mechanism for edge-cover, hence, we chose to restrict our attention to two possible prices. Even though, one could construct fully budget-balanced group-strategyproof mechanisms that use more than two prices, their analysis would be even more complicated. Our goal is to show the existence of such a mechanism rather characterize them, hence, we restricted to something intuitive and simple. Moreover, it allowed us to implement the allocation of the mechanism in polynomial time, whereas it is not certain that this would be possible if the cost-sharing scheme was more complicated. It is open whether natural and simple cost-sharing schemes for other interesting problems happen to satisfy the sufficient conditions [14] and in that case if we can find efficient implementation.

We also show that not all problems have fully budget-balanced group-strategyproof mechanisms. This is fairly obvious when one allows arbitrary (e.g., non-monotone) cost functions.[2] In this paper, we prove this result for the natural monotone cost function defined by the set cover problem, a generalization of the edge cover problem. For set cover, there is a bound of $n^{-1/2}$ (where n is the number of agents) on the budget-balance factor of cross-monotonic cost-sharing schemes [6], implying that the standard technique for designing group-strategyproof mechanisms is highly impractical. This negative result is particularly disturbing in light of the fact that there exists a trivial fully budget balanced strategyproof mechanism (see Example 4.1 [6]) for any non-decreasing cost-function if we don't take computational limits into consideration. Even imposing computational limits, we can obtain a $O(1/\log n)$-budget balanced mechanism that is strategyproof but not group-strategyproof [3]. In our work we are interested in bounding the power of group-strategyproofness without any computational assumption. We present a set-cover instance, where there is no cost-sharing scheme satisfying the characterization of Pountourakis and Vidali [14] with budget-balance factor better than $18/19$. Since this characterization does not take computational constraints into consideration this implies a bound for every group-strategyproof mechanism independent of its running time.

Finally, we would like to note that while we try to deal with the limitations of cross-monotonic mechanisms by exploiting the full power of group-strategyproofness, another approach that has been followed so far was to relax group-strategyproofness. In particular, there is a general framework to design weak group-strategyproonf mechanisms [9]. This framework has been used

[2] See, for example [14].

to design mechanism with better budget balance guarantees for many combinatorial problems [9, 1]. Specifically, cost-sharing schemes used by [9] were naturally derived by primal-dual schemes without any refinement that would ensure cross-monotonicity [12]. However, as we argue in Section 5 these cost-sharing schemes fail to satisfy the necessary condition to give rise to group-strategyproofness; hence, ideas from this literature cannot be directly applied to group-strategyproof mechanism design.

Related Work In addition to the literature on cost-sharing mechanisms mentioned above, our work is related to the literature on combinatorial public projects. This problem was introduced in [13] who assume that a set of agents is interested in sharing a number of resources. Each agent has a private valuation for each subset of these resources. For some given k, a mechanism has to choose based on the valuations of the agent a set of k resources so as to maximize the social welfare.

There are communication and computational bounds for every strategyproof mechanism that solves this problem when the valuation functions satisfy submodularity [13]. There are similar bounds without the constraint of truthfulness, but slightly relaxing submodularity of valuation functions [15]. Finally, recent work [2] studies similar questions for sub-additive valuations and provide various upper and lower bound for specific valuation function classes.

This problem differs from cost-sharing in the sense that there are multiple resources the mechanism is called to choose upon, however, all the agents are going to share them. Moreover, they are interested in maximizing social welfare rather than budget balance. However, both of these problems have applications to resource sharing and particularly network formation.

2 Model

A set of agents $\mathcal{A} = \{1, 2, \ldots, n\}$ is interested in receiving a service. Each agent i has a private type v_i, which is her valuation for receiving the service. A *cost-sharing mechanism* inputs a bid b_i for each agent i and outputs the subset of agents $Q \subseteq \mathcal{A}$ that receive service and the price p_i that each agent i pays. Assuming quasi-linear utilities, each agent wants to maximize the quantity $v_i x_i - p_i$ where $x_i = 1$ if $i \in Q$ and $x_i = 0$ if $i \notin Q$. We concentrate on mechanisms that satisfy the following conditions [10, 11]:

- *Voluntary Participation (VP)*: An agent that is not serviced is not charged ($i \notin Q \Rightarrow p_i = 0$) and a serviced agent is never charged more than her bid ($i \in Q \Rightarrow p_i \leq b_i$).
- *No Positive Transfer (NPT)*: The payment of each agent i is non-negative ($p_i \geq 0$ for all i).
- *Consumer Sovereignty (CS)*: For each agent i there exists a value $b_i^* \in \mathbb{R}$ such that if she bids b_i^*, then it is guaranteed that agent i will receive the service no matter what the other agents bid.

We also assume that the agents can bid in a way that they will definitely not receive the service. This can be done by allowing negative bids. Then VP implies that an agent that reports a negative amount has to be charged a negative amount if she is serviced, which is prohibited by NPT.

We are interested in mechanisms that are *group-strategyproof (GSP)*. A mechanism is GSP if for every two valuation vectors v, v' and every coalition of agents $S \subseteq \mathcal{A}$, satisfying $v_i = v'_i$ for all $i \notin S$, one of the following is true: *(a)* There is some $i \in S$, such that $v_i x'_i - p'_i < v_i x_i - p_i$ or *(b)* for all $i \in S$, it holds that $v_i x'_i - p'_i = v_i x_i - p_i$, where x'_i and p'_i is the allocation and payment of player i respectively when the agents report v'.

We also assume the existence of an underlying cost-function $C : 2^{\mathcal{A}} \to \mathbb{R}^+ \cup \{0\}$, where $C(S)$ specifies the cost of providing service to all agents in S. We say that a mechanism is *α-budget balanced* with respect to C if for all b, $\alpha C(Q) \leq \sum_{i \in \mathcal{A}} p_i \leq C(Q)$, where Q and $\{p_i\}$ are the prices and allocation output by the mechanism on input b.

2.1 Characterization

A cost-sharing scheme $\xi : \mathcal{A} \times 2^{\mathcal{A}} \to \mathbb{R}$ is a function that takes as input an agent and a set and outputs a real number. The amount $\xi(i, S)$ can be viewed as the payment of agent i when the set of agents S receives the service.[3] A cost-sharing scheme is *α-budget balanced* with respect to C if for all S, $\alpha C(S) \leq \sum_{i \in S} \xi(i, S) \leq C(S)$.

A property of the cost-sharing scheme, namely *cross-monotonicity*, has played a central role in the literature. Intuitively, cross-monotonicity requires that the cost-share of an agent can not increase as the serviced set grows. Moulin [10, 11] showed that given a cross-monotone cost-sharing scheme we can construct a group-strategyproof mechanism with a budget-balance factor equal to that of the cost-sharing scheme. Moreover, if the underlying cost function is submodular then there exist a perfectly budget balanced cost-sharing scheme. However, when the cost function is given by the cost of the optimal solution of an optimization problem, the cost function is often not submodular. Subsequent work [6] proved bounds on the budget balance factor of cross-monotonic cost-sharing schemes. This gave rise to the question of whether there are group-strategyproof mechanisms for such problems with better budget-balance properties. A step towards answering this question was taken by [14], where they gave a complete characterization of the cost-sharing schemes that can give rise to group-strategyproof mechanisms. Let $\xi^*(i, L, U)$ be the minimum payment of player i for getting serviced when the serviced set is "between" sets L and U, i.e., $\xi^*(i, L, U) := \min_{\{L \subseteq S \subseteq U, i \ni S\}} \xi(i, S)$.

[3] This is a restrictive form of a payment policy as we exclude the possibility of charging different values given two different bid vectors where the mechanism provides the service to the same set of agents. Nevertheless, this is without loss of generality for the mechanisms of our setting [6].

Theorem 1 (Pountourakis and Vidali [14]). *A cost-sharing scheme ξ can give rise to a group-strategyproof mechanism if and only if for every $L \subseteq U \subseteq \mathcal{A}$ it satisfies the following three properties.*

(a) There exists a set S with $L \subseteq S \subseteq U$, such that for all $i \in S$, we have $\xi(i, S) = \xi^*(i, L, U)$.
(b) For each player $i \in U \setminus L$ there exists one set S_i with $i \in S_i$ and $L \subseteq S_i \subseteq U$, such that for all $j \in S_i \setminus L$, we have $\xi(j, S_i) = \xi^*(j, L, U)$.
(Since $i \in S_i \setminus L$, it holds that $\xi(i, S_i) = \xi^*(i, L, U)$.)
(c) If for some $C \subset U$ there is a player $j \in C$ with $\xi(j, C) < \xi^*(j, L, U)$ (obviously $L \nsubseteq C$), then there exists a set $T \neq \emptyset$ with $T \subseteq L \setminus C$, such that for all $i \in T$, $\xi(i, C \cup T) = \xi^*(i, L, U)$.

2.2 Allocation

Additionally, there is a complete characterization of the allocation functions that can be coupled with a cost-sharing scheme satisfying the properties of Theorem 1 to yield a group-strategyproof mechanism [14] .

Theorem 2 (Pountourakis and Vidali [14]). *If ξ is a cost-sharing scheme that satisfies the properties of Theorem 1, then for every bid vector b there exist unique sets $L(b) \subseteq U(b) \subseteq \mathcal{A}$ such that*

1. *for all $i \in L(b)$, $b_i > \xi^*(i, L(b), U(b))$,*
2. *for all $i \in U(b) \setminus L(b)$, $b_i = \xi^*(i, L(b), U(b))$,*
3. *and for all $R \subseteq \mathcal{A} \setminus U(b)$, there exist $i \in R$ with $b_i < \xi^*(i, L(b), U(b) \cup R)$.*

Furthermore, the mechanism that on input b outputs allocation $Q = S$ and prices $p_i = \xi(i, S)$ where

1. *$L(b) \subseteq S \subseteq U(b)$, and*
2. *for all $i \in S$, $\xi(i, S) = \xi^*(i, L(b), U(b))$ (such a set must exist by Theorem 1(a)),*

is a group-strategyproof mechanism.

A way to implement the allocation is to exhaustively search for these sets $L(b)$ and $U(b)$ and a set S that satisfy the properties of the theorem. It is still not known if there is an algorithm that implements this procedure with asymptotically better running time in general. However, in Section 3, we provide a fully budget-balanced cost-sharing scheme and corresponding polynomial-time allocation when the cost function is given by the edge-cover problem.

3 Edge Cover

In this section, we give a fully budget-balanced group-strategyproof mechanism for the unweighted edge-cover cost-sharing game. To do so, we derive a cost-sharing scheme that satisfies the conditions presented in Theorem 1. Previous work [6] implies that group-strategyproof mechanisms for edge-cover designed via cross-monotonic cost-sharing schemes are at best 1/2-budget balanced. Thus, our work improves upon the previous results and demonstrates that the assumption of cross-monotonicity is not without loss for group-strategyproof mechanism design. We start with a definition of the edge cover game.

Definition 1. *In the* edge cover game *we are given a graph $G = (V, E)$ with no isolated vertices. The agents in the game are the vertices V of the graph G. Given a subset of vertices $S \subseteq V$, an edge-cover of S is a subset of edges $F \subseteq E$ such that for all vertices $v \in S$ there is some edge $e \in F$ such that $v \in e$. The cost of a set S is the minimum cardinality edge-cover of S.*

In the following subsections, we first present a cost-sharing scheme for the edge cover game that provably gives rise to a group-strategyproof mechanism. We then show how to use this scheme to define a computationally efficient group-strategyproof mechanism for our problem.

3.1 Cost-sharing Scheme

Our cost-sharing scheme is based on the following well-known polynomial time algorithm [4] for finding the minimum edge cover F of a set S. Let F be the set of edges in the maximum matching on S, and then for each vertex $v \in S$ uncovered by F, add to F an edge e adjacent to v. Based on this algorithm, a natural cost-sharing scheme is to charge each agent $v \in S$ a price of $1/2$ if v is in the matching found by the algorithm, and 1 otherwise. The problem that arises with this cost-sharing scheme is the existence of multiple maximum matchings. We demonstrate this in the appendix using an example of bad tie-breaking rule among maximum matchings.

Definition 2. *Given $G = (V, E)$ on m edges, label edges E from 1 to m arbitrarily. For a subset of vertices $S \subseteq V$, let M_S denote the lexicographically first maximum matching according to the labeling. Moreover let $V(M_S) = \{v \mid \exists e \in M_S \text{ s.t. } v \in e\}$, i.e., $V(M_S)$ contains the vertices that are matched in M_S.*

Note that the lexicographically first maximum matching M_S of any set of vertices S can be found efficiently, for example by assigning a weight of $(1 + 2^i)/2^m$ to the i'th edge and then computing the maximum weight matching. We are now ready to formally define the cost-sharing scheme ξ. This cost-sharing scheme extends one introduced by [6] for an edge-cover instance on just three vertices as an example of a group-strategyproof mechanism without a cross-monotone cost-sharing scheme.

Definition 3. *Let $G = (V, E)$. For every $S \subseteq V$ and every $i \in V$ we define*

$$\xi(i, S) = \begin{cases} 0 & i \notin S \\ 1/2 & i \in V(M_S) \\ 1 & i \in S \setminus V(M_S) \end{cases}$$

By construction, the cost-sharing scheme of Definition 3 is 1-budget balanced (and therefore, by the results of [6], it is not cross-monotone). We show that it additionally satisfies all the necessary and sufficient conditions of group-strategyproofness.

Theorem 3. *The cost-sharing scheme ξ of Definition 3 satisfies all the necessary and sufficient conditions to give rise to a GSP mechanism. Consequently there is a* 1*-budget balanced GSP mechanism for the edge-cover problem.*

The proof uses the characterization presented in Theorem 1. The main challenge is to show that for any lower set L and upper set U, there is some intermediate set $S^*, L \subseteq S^* \subseteq U$, in which every agent in S^* achieves his minimum cost-share among all intermediate sets (i.e., property (a)). Since cost-shares are always either 1 or 1/2, this amounts to finding a set S^* in which each agent $i \in S^*$ either has cost-share 1/2, or has cost-share 1 for every intermediate set $S, L \subseteq S \subseteq U$.

The proof idea is as follows. We start with an arbitrary intermediate set S and work our way towards S^*. First, we prove in the following lemmas that for any set S, we can discard agents with cost-share equal to 1 without changing the solution for the other agents. Thus starting from an arbitrary intermediate set S, we can work our way towards S^* by discarding all agents in $S \setminus L$ with cost-share equal to 1. This leaves the question of whether agents in $S \cap L$ are receiving their minimum cost-share among the intermediate sets. Unfortunately, this is not necessarily the case: there may be some unhappy agent $i \in L \cap S$ with cost-share equal to 1 who has a cost-share equal to 1/2 in some other intermediate set S_i. In this case, we use the lexicographically first maximum matchings M_S and M_{S_i} to construct an alternating path starting from agent i. We prove that this alternating path ends in a node j that can either be added to or deleted from S in order to decrease the number of unhappy agents in $L \cap S$ (interestingly, agent i may still be unhappy after this fix, but at least one agent becomes happy). In this way, starting from an arbitrary intermediate set S, we can walk towards S^*. We defer the full proof to the appendix.

3.2 Polynomial-Time Allocation

We now argue that the group-strategyproof mechanism corresponding to the cost-sharing scheme in Definition 3 can be constructed in polynomial time. To do so, we must find, for any bid vector b, a set S satisfying the conditions of Theorem 2. Namely, we are looking for a set S that lies between some lower-bound set $L(b)$ and upper-bound set $U(b)$ such that:

1. for all $i \in L(b)$, $b_i > \xi^*(i, L(b), U(b))$,
2. for all $i \in U(b) \setminus L(b)$, $b_i = \xi^*(i, L(b), U(b))$,
3. and for all $R \subseteq \mathcal{A} \setminus U(b)$, there exist $i \in R$ with $b_i < \xi^*(i, L(b), U(b) \cup R)$,

and for all $i \in S$, $\xi(i, S) = \xi^*(i, L(b), U(b))$. In words, the elements in $L(b)$ should be bidding more than their minimum cost-share; the elements in $U(b) \setminus L(b)$

should be bidding equal to their minimum cost-share; and $U(b)$ is maximal in the sense that when we try to add elements to it, at least one of the newcomers can't afford his minimum cost-share. The set S allocated by the group-strategyproof mechanism is then any of the intermediate sets in which each agent is happy (gets its minimum cost-share), i.e., a set S with

$$L(b) \subseteq S \subseteq U(b), s.t.\ \forall i \in S, \xi(i, S) = \xi^*(i, L(b), U(b)).$$

For ease of notation, in the rest of this section we fix b and use L to denote $L(b)$ and U to denote $U(b)$.

The main difficulty in finding S is that we do not know L and U. However, using the structure of these sets and the fact that the only cost-shares in our scheme are 1 and 1/2, we can bound these two sets. Given a bid vector b let us define $P = \{i \mid b_i > \frac{1}{2}\}$ and $Q = \{i \mid b_i \geq \frac{1}{2}\}$. Then $L \subseteq P$ and $U \subseteq Q$. Hence we can search through intermediate sets of P and Q, looking for our S. Any such S will definitely contain L as $L \subseteq P$, but may not be contained in U; our algorithm must provide a separate guarantee for this containment.

Our algorithm for finding S is based on a local search procedure and corresponding potential function $\phi(.)$ which is strictly increasing with respect to the steps of this search. The search procedure iteratively adds an element to, or deletes an element from, the current set S while maintaining the invariant that $P \subseteq S \subseteq Q$. Our potential function $\phi(S)$ counts the number of happy elements in $L \subseteq S$, i.e.,

$$\phi(S) = |\{i \in L \mid \xi(i, S) = \xi^*(i, L, U)\}|.$$

We show that as long as $\phi(S) < |L|$, there is always a way to improve the potential. Since L is fixed and finite (given b), this procedure must terminate. Furthermore, by definition of the potential, when the procedure terminates, each agent in $L \subseteq S$ is happy. To guarantee that agents in $S \setminus L$ are happy and also that $S \subseteq U$, we need to prune S. As we show later, it is sufficient to simply remove agents from S whose bids are less than their cost-shares. The following procedure implements this local search.

1. $S \leftarrow P$.
2. Iterate as long as the set S changes:
   (a) Remove all players in $S \setminus P$ with $\xi(i, S) = 1$.
   (b) If there is some $i \in Q \setminus S$ such that the cardinality of the maximum matching in $S \cup \{i\}$ is increased, then set $S \leftarrow S \cup \{i\}$.
   (c) Else if there is some $i \in S \setminus P$ that was matched in $M(S)$ and the maximum matching in $S \setminus \{i\}$ does not decrease, then set $S \leftarrow S \setminus \{i\}$.
3. Set $S \leftarrow \{i \mid b_i \geq \xi(i, S)\}$.

We first observe that this algorithm runs in polynomial time. Specifically steps 2 (b) and 2 (c) reduce to finding whether the inclusion of some agent in $Q \setminus P$ forms an augmented path or whether an agent in $S \setminus P$ is not present in every maximum cardinality matching respectively. Both of these steps can be implemented in polynomial time. Since at each step the potential function increases, step 2 is performed at most as many times as the cardinality of L, which is bounded by the total number of agents. The full proof can be found in the appendix.

Theorem 4. *This procedure outputs a set S, $L \subseteq S \subseteq U$, where for all $i \in S$, $\xi(i, S) = \xi^*(i, L, U)$.*

4 Set Cover

In this section we show that it is impossible to construct a fully budget balanced group-strategyproof mechanism when the cost function is determined by the optimal objective function of the set-cover problem. It is known that no cross-monotonic cost-sharing scheme can have a budget-balance of better than $n^{-1/2}$ where n is the number of elements or the size of the largest subset in the set-cover instance [6]. Here we show that there are instances where no group-strategyproof mechanism can be (18/19)-budget-balanced. Thus, while group-strategyproof mechanisms may be able to improve upon the budget-balance factor of cross-monotonic ones, we show that they can not, in general, provide full budget-balance.

Definition 4. *In the* set cover game *we are given a ground set V and a collection of subsets $\mathcal{F} \subseteq 2^V$. The agents in the game are the elements V of the ground set. Given a subset of agents $S \subseteq V$, a set-cover of S is a collection of subsets $\mathcal{C} \subseteq \mathcal{F}$ such that $S \subseteq \bigcup_{C \in \mathcal{C}} C$, i.e., every element $i \in S$ belongs to some subset $C \in \mathcal{C}$. The cost of a set of agents S is the minimum cardinality set-cover of S.*

In the following subsection, in order to build intuition, we first prove that there is no fully budget-balanced group-strategyproof mechanism for the set-cover game. Our counter-example uses the following instance of a set-cover game. There are six elements $U = \{A_1, A_2, B_1, B_2, C_1, C_2\}$, and the collection of subsets is $\mathcal{F} = \{\{A_i, B_j, C_k\}_{i,j,k=1,2}\}$. In other words there are three groups of two elements and the available subsets are all those who contain exactly one element from each group. We then extend the proof to show the constant bound.

4.1 Impossibility of Full Budget-Balance

We will show by contradiction that there is no fully budget balanced group-strategyproof mechanism for this instance of the set-cover game. The first step to reach a contradiction is to show that the necessary properties together with full budget balance imply that at sets of the form $\{A_1, A_2, B_j, C_k\}$, there must be an unfair sharing of the cost in the sense that A_1 or A_2 must be responsible for their externality (their inclusion increases the cost of the optimal solution by one). Since one of the agents of the group A is responsible for her externality this puts an upper bound on the sum of the rest agents' payments. Then, since adding the missing agent from group B does not increase the cost, we show this does not change the upper bound. Finally, we exploit the symmetric form of this instance to derive the same bound with different agents in groups A and B. By summing, we deduce that the contribution of agent C_k is zero in the set U. Applying the same argument for every agent we deduce that no agent must be charged in U reaching a contradiction.

Theorem 5. *There is no fully budget-balanced group-strategyproof mechanism for the set-cover game.*

A slight different approach can be used to obtain a constant lower bound for this example.

Theorem 6. *There is no (18/19)-budget-balanced group-strategyproof mechanism for set cover.*

We refer the reader to the appendix of the paper for the full proofs of these theorems.

5 Conclusion

Our work is the first application of the complete characterization of group-strategyproof mechanisms [14]. We use the characterization to show bounds on the budget balance of group-strategyproof mechanisms for specific combinatorial problems. Particularly, we show that a very natural cost-sharing scheme for edge-cover satisfies the conditions of the characterization and is fully budget balanced. While the case of edge-cover is completely solved by this paper, it remains open to bound the optimal budget balance factor of group-strategyproof mechanisms for set-cover. Other problems of interest include facility location, vertex cover, Steiner tree, and Steiner forest. In the previous literature, these problems have only been solved using techniques involving cross-monotonic cost-sharing schemes, and it is known such an approach can not achieve perfect budget-balance for these problems.

Many constructions of cross-monotonic cost-sharing schemes rely on primal-dual schema. In these schemes, the natural linear-programming formulations of the combinatorial problems have constraints corresponding to the demand of the agents. The primal-dual scheme charges each agent her respective dual variable. In order to guarantee cross-monotonicity, these schemes introduce the notion of *ghost-shares*, i.e., the idea that variables contributing to a tight constraint are not frozen but rather keep growing and contributing to other constraints. Nevertheless, the payment of an agent is determined by the time that the dual variable was first involved in a tight constraint.

If we don't use the ghost-share technique the resulting cost-sharing scheme is not cross-monotone. However, in many cases it can be used to design weakly group-strategyproof mechanisms [9]. A natural question that arises is whether such a cost-sharing scheme satisfies the necessary conditions of group-strategyproofness despite the fact that it fails to satisfy cross-monotonicity. Unfortunately, the following observation indicates that this may not be true. Consider a cost-sharing scheme that is constructed by a primal-dual scheme and does not satisfy cross-monotonicity. Note that this implies the existence of a set S and two agents $i, j \in S$ such that $\xi(j, S \setminus \{i\}) < \xi(j, S)$. This means that the constraint that was responsible for freezing the dual variable of agent j becomes tight at a later time when i is present. This is only possible if there is another agent $k \in S$ that contributed to this constraint for subset $S \setminus \{i\}$; however, the inclusion of agent i caused the variable of k to freeze earlier, which means that $\xi(k, S \setminus \{i\}) > \xi(k, S)$. Such a cost-sharing scheme would not satisfy even the weaker necessary property of semi-cross monotonicity identified in [6].

The previous observation indicates that one should search beyond primal-dual schemes in order to design group-strategyproof mechanisms that perform strictly better than mechanisms captured by the cross-monotonic framework.

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