

# A complete characterization of group-strategyproof mechanisms of cost-sharing<sup>\*</sup>

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**Abstract.** We study the problem of designing group-strategyproof cost-sharing mechanisms. The players report their bids for getting serviced and the mechanism decides a set of players that are going to be serviced and how much each one of them is going to pay. We determine three conditions: *Fence Monotonicity*, *Stability* of the allocation and *Validity* of the tie-breaking rule that are necessary and sufficient for group-strategyproofness, regardless of the cost function. Consequently, Fence Monotonicity characterizes group-strategyproof cost-sharing schemes closing an important open problem. Finally, we use our results to prove that there exist families of cost functions, where any group-strategyproof mechanism has arbitrarily poor budget balance.

## 1 Introduction

Algorithmic Mechanism Design [1] is a field of Game Theory, that tries to construct algorithms for allocating resources that give to the players incentives to report their true interest in receiving a good, a service, or in participating in a given collective activity. The pivotal constraint when designing a mechanism for any problem is that it is truthful. Truthfulness also known as strategyproofness requires that no player can strictly improve her utility by lying. In many settings this single requirement for an algorithm to be truthful restricts the repertoire of possible algorithms dramatically [2–5].

In settings, where truthfulness does not impose such severe limitations, like for example in Cost-sharing problems, it is desirable to construct mechanisms that are also resistant to manipulation by groups of players. A group of players forms a *successful coalition* when the utility of *each* player in the group does not decrease and the utility of *at least one* player *strictly* increases. Group-strategyproofness naturally generalizes truthfulness by requiring that no group of players can form a successful coalition by lying, when the values of the other players are fixed.

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In this paper we study the following problem: A set of  $n$  customers/players are interested in receiving a service. Players report their valuation of the service and the mechanism decides the players that are going to be serviced and the price that each one of them will pay, i.e., we consider direct revelation mechanisms. We want to characterize all possible mechanisms that satisfy group-strategyproofness via identifying necessary and sufficient conditions for the payment functions that give rise to these mechanisms.

We provide a complete characterization of group-strategyproof mechanisms closing an important open question posed in [6], [7, Chapter 15]. Immorlica, Mahdian and Mirrokni [6] identified the property of semi-cross-monotonicity, a local property that should be satisfied by the payment part (cost-sharing scheme) of every group-strategyproof mechanism. We introduce a generalization of semi-cross-monotonicity, Fence Monotonicity, which still refers only to the payment part of the mechanism, and is not only *necessary* but also *sufficient* for group-strategyproofness. Given any payment rule that satisfies Fence Monotonicity, we show that every allocation that satisfies *Stability* and *Validity* of the tie-breaking rule, yields a group-strategyproof mechanism. Our characterization of group-strategyproofness is regardless of the cost function and without any additional constraints (like fixed tie-breaking rules [6, 8]).

Our results provide a new, as general as it can be, framework for designing group-strategyproof mechanisms. Thus, it opens new perspectives to the study of the very important problem of Cost-Sharing, the study of which was initiated by Moulin [9], where we additionally have a cost function  $C$  such that for each subset of players  $S$  the cost for providing service to all the players in  $S$  is  $C(S)$ . However, the strength of our results is that they apply to any cost function, since throughout our proof we do not make any assumptions at all about this cost function. We believe that our work here can be the starting point for constructing new interesting classes of mechanisms for specific cost-sharing problems.

As our characterization is the first characterization of group-strategyproofness, it is interesting to make a parallelization between the many known characterizations of strategyproofness [10, 11, 2–5] and our characterization of group-strategyproofness, not however, in the technical level, but in order to determine the quality and usefulness of our characterization. Fence Monotonicity is a condition rather similar to Cycle Monotonicity [10], in the sense that both are conditions that should be satisfied by all possible restrictions of the output space of the mechanism. A great virtue for a characterization of a mechanism that uses money is to be able to separate the payment from the allocation part. Cycle Monotonicity is a necessary property for the allocation. If it is satisfied, we know a way to define truthful payments [10]. Fence Monotonicity refers only to the payments of a mechanism and if it is satisfied, Stability and Validity of the tie-breaking rule can be used to determine a group-strategyproof allocation. Managing to separate the payments from the allocation and avoiding to add any additional restrictions in the characterization we propose are undoubtedly its great virtues.

We believe that Fence Monotonicity can also be a starting point in the quest for alternative characterizations, as Cycle monotonicity has done so far [11, 2–5]. Unfortunately, even though Fence Monotonicity is succinct in its description, it is much more complicated than Cycle Monotonicity. Nonetheless, group-strategyproofness is a notion much more complicated than strategyproofness and we believe that Fence Monotonicity is not only important and unavoidable but also useful. We demonstrate the latter by proving the first of its kind very simple lower bound on the budget balance of any group-strategyproof mechanism.

### 1.1 Our results and related work.

The design of group-strategyproof mechanisms for cost-sharing was first discussed by Moulin and Shenker [9, 12]. Moulin [9] defined a condition on the payments called cross-monotonicity, which states that the payment of a serviced player should not increase as the set of serviced players grows. Any mechanism with payments satisfying cross-monotonicity can be easily turned to a simple mechanism called after Moulin. A Moulin mechanism first checks if it can provide service to all players, so that each one has non-negative utility and if not, it gradually diminishes the set of players that are candidates to be serviced, by throwing away at each step a player that cannot pay to get serviced (and who because of cross-monotonicity still cannot pay if the set of candidates becomes smaller). In fact, if the cost function is sub-modular and 1-budget balanced, then the only possible group-strategyproof mechanisms are Moulin mechanisms [12]. The great majority of cost-sharing mechanisms proposed are Moulin mechanisms.

Nevertheless, recent results showed that for several important cost-sharing games Moulin mechanisms can only achieve a very bad budget balance factor [6, 13, 14]. Some alternative, very interesting, and much more complicated in their description mechanisms that are group-strategyproof but not Moulin have been proposed [6, 15], however, these do not exhaust the class of group-strategyproof mechanisms. In this work we introduce *Fencing Mechanisms*, a new general framework for designing group-strategyproof mechanisms, that generalizes Moulin mechanisms [9].

Recently, Mehta et. al. [16] proposed the notion of weak group-strategyproofness, that relaxes group-strategyproofness. It regards a formation of a coalition, as successful, when *each* player who participates in the coalition *strictly* increases her personal utility. They also introduce acyclic mechanisms, a general framework for designing weakly group-strategyproof mechanisms, however, the question of determining all possible weakly group-strategyproof mechanisms is an important question that remains open. Another alternative notion was proposed by Bleischwitz et. al. in [17].

The problem we solve here was a major open problem posed in [6],[7, Chapter 15]. Many very interesting results arose in the attempt to find such a characterization [15, 8]. In contrast to previous characterization attempts that characterized mechanisms satisfying some additional boundary constraints [6, 8], our characterization is complete and succinct. The only previously known complete

characterization was for the case of two players [15, 8]. The great challenge now is how our characterization can be applied for constructing new efficient mechanisms for specific cost-sharing problems. We believe that it can significantly enrich the repertoire of mechanisms with good budget balance guarantees for specific problems.

In the notion of group-strategyproofness it is important to understand that ties play a very important role. This is in contrast to strategyproofness, where ties can be in most cases broken arbitrarily (see for example [18]). An intuitive way to understand this is that a mechanism designer of a group-strategyproof mechanism expects a player to tell a lie in order to help the other players increase their utility, even when she would not gain any profit for herself. This player is at a tie but decides strategically if she should lie or not. Consequently, a characterization that assumes a priori a tie-breaking rule, and hence, greatly restricts the repertoire of possible mechanisms [6, 19] might be useful for specific problems and easier in its statement, but can never capture the very notion group-strategyproofness.

Our proofs are involved and make a repeated use of induction. The main difficulty and value of our work though, was to identify necessary and sufficient conditions for group-strategyproof payments, that are also succinct to describe and add to our understanding of the notion of group-strategyproofness. In proving the necessity of Fence Monotonicity we first have to prove lemmas that also reveal interesting properties of the allocation part of the mechanism. A novel tool that we introduce is the *harm relation* that refines the notion of negative elements defined in [6]. Proving that Fencing Mechanisms, i.e., mechanisms with payments satisfying Fence Monotonicity and allocation satisfying Stability and Validity of the tie-breaking rule, are group-strategyproof turns out to be rather complicated. It is likely that our techniques can be extended for characterizing weakly group-strategyproof mechanisms.

## 2 Defining the model

**The Mechanism** Suppose that  $\mathcal{A} = \{1, 2, \dots, n\}$  is a set of players interested in receiving a service. Each of the players has a private type  $v_i$ , which is her valuation for receiving the service.

A cost sharing mechanism  $(O, p)$  consists of a pair of functions,  $O : \mathbb{R}^n \rightarrow 2^{\mathcal{A}}$  that associates with each bid vector  $b$  the set of serviced players and  $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that associates with each bid vector  $b$  a vector  $p(b) = (p_1(b), \dots, p_n(b))$ , where the  $i$ -th coordinate is the payment of player  $i$ . Assuming quasi-linear utilities, each player wants to maximize the quantity  $v_i a_i - p_i(b)$  where  $a_i = 1$  if  $i \in O(b)$  and  $a_i = 0$  if  $i \notin O(b)$ .

As it is common in the literature and in order to avoid absurd mechanisms, we concentrate on mechanisms that satisfy the following very simple conditions [9, 12, 6]:

- *Voluntary Participation (VP)*: A player that is not serviced is not charged ( $i \notin O(b) \Rightarrow p_i(b) = 0$ ) and a serviced player is never charged more than her bid ( $i \in O(b) \Rightarrow p_i(b) \leq b_i$ ).
- *No Positive Transfer (NPT)*: The payment of each player  $i$  is non-negative ( $p_i(b) \geq 0$  for all  $i$ ).
- *Consumer Sovereignty (CS)*: For each player  $i$  there exists a value  $b_i^* \in \mathbb{R}$  such that if she bids  $b_i^*$ , then it is guaranteed that player  $i$  will receive the service no matter what the other players bid.<sup>3</sup>

In accordance with [9, 6] we will concentrate on mechanisms, where the players are additionally able to deny receiving service, no matter what the other players bid. To ensure this we just assume that players can report negative bids. Then VP and NPT imply that if a player announces a negative amount, she will not receive the service. Even though negative bids may not seem realistic, they can model the denial of revealing any information to the mechanism. To put things simply, if a player  $i$  bids  $-1$ , then she does not receive service and if she bids  $b_i^*$ , then she receives service, no matter what the other players bid.

We are interested in mechanisms that are *group-strategyproof (GSP)*. A mechanism is GSP if for every two valuation vectors  $v, v'$  and every coalition of players  $S \subseteq \mathcal{A}$ , satisfying  $v_i = v'_i$  for all  $i \notin S$ , one of the following is true: (a) There is some  $i \in S$ , such that  $v_i a'_i - p_i(v') < v_i a_i - p_i(v)$  or (b) for all  $i \in S$ , it holds that  $v_i a'_i - p_i(v') = v_i a_i - p_i(v)$ .

**The cost function, cost sharing scheme, and budget balance.** The cost of providing service is given by a cost function  $C : 2^{\mathcal{A}} \rightarrow \mathbb{R}^+ \cup \{0\}$ , where  $C(S)$  specifies the cost of providing service to all players in  $S$ .

A *cost-sharing scheme* is a function  $\xi : \mathcal{A} \times 2^{\mathcal{A}} \rightarrow \mathbb{R}^+ \cup \{0\}$  such that for every  $S \subset \mathcal{A}$  and every  $i \notin S$ , we have  $\xi(i, S) = 0$ . Immorlica et. al. [6] showed that in any GSP mechanism of our setting the payment of a player depends only on the allocation produced by the mechanism for a particular bid vector and not directly on the bids of the players. As a consequence, the payments of any GSP mechanism are given by a cost-sharing scheme  $\xi$ , i.e.,  $p_i(b) = \xi(i, O(b))$ .

A desirable property cost-sharing schemes with respect to some cost function is budget balance. We say that a cost-sharing scheme  $\xi$  is  $\alpha$ -*budget balanced*, where  $0 \leq \alpha \leq 1$ , if for  $S \subseteq \mathcal{A}$  it holds that  $\alpha \cdot C(S) \leq \sum_{i \in S} \xi(i, S) \leq C(S)$ . Correspondingly, we say that a mechanism is  $\alpha$ -budget balanced if its cost sharing scheme is  $\alpha$ -budget balanced. We chose to define the cost function last in order to stress that our results are completely independent of the cost function and any budget-balance assumption, thus they apply to any cost-sharing problem.

An important and well-studied property of cost-sharing schemes is cross-monotonicity, which is sufficient for group-strategyproofness [9]. A cost-sharing

<sup>3</sup> From VP it holds that  $b_i^* \geq \max_{b \in \mathbb{R}^n} p_i(b)$ . It is easy to verify that strategyproofness implies that any value greater than  $b_i^*$  satisfies CS for player  $i$ . Thus, when we refer to this crucial value, we will without loss of generality assume that this inequality is strict.

scheme is *cross-monotonic* if  $\xi(i, S) \geq \xi(i, T)$  for every  $S \subset T \subseteq \mathcal{A}$  and every player  $i \in S$ . This means that the payment of a player cannot increase as the number of players that receive service increases.

In the attempt to provide a characterization of GSP mechanisms Immorlica et. al. [6] provided a partial characterization and identified semi-cross-monotonicity, an important condition that should be satisfied by the cost-sharing scheme of any GSP mechanism. A cost sharing scheme  $\xi$  is *semi-cross-monotonic* if for every  $S \subseteq \mathcal{A}$ , and every player  $i \in S$ , either for all  $j \in S \setminus \{i\}$ ,  $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$  or for all  $j \in S \setminus \{i\}$ ,  $\xi(j, S \setminus \{i\}) \geq \xi(j, S)$ .

As we later show in Proposition 1 (i), semi-cross-monotonicity can be almost directly derived by Fence Monotonicity, the property we introduce in this work.

### 3 Our Characterization

#### 3.1 Fence Monotonicity

Fence Monotonicity considers each time a restriction of the mechanism that can only output as the serviced set, subsets of  $U$  that contain all players in  $L$ . To be more formal, consider all possible subsets of the players  $L, U$  such that  $L \subseteq U \subseteq \mathcal{A}$ . Given a pair  $L \subseteq U$ , Fence Monotonicity considers only sets of players  $S$  with  $L \subseteq S \subseteq U$ . Each Fence Monotonicity condition should be satisfied for any possible such restriction of output space of the mechanism.

We denote by  $\xi^*(i, L, U)$  the minimum payment of player  $i$  for getting serviced when the output of the mechanism is restricted by  $L$  and  $U$ , i.e.,  $\xi^*(i, L, U) := \min_{\{L \subseteq S \subseteq U, i \in S\}} \xi(i, S)$ .

**Definition 1 (Fence Monotonicity).** *We will say that a cost-sharing scheme satisfies Fence Monotonicity, if for every  $L \subseteq U \subseteq \mathcal{A}$ , it satisfies the following three conditions:*

- (a) *There exists at least one set  $S$  with  $L \subseteq S \subseteq U$ , such that for all  $i \in S$ , we have  $\xi(i, S) = \xi^*(i, L, U)$ .*
- (b) *For each player  $i \in U \setminus L$  there exists at least one set  $S_i$  with  $i \in S_i$  and  $L \subseteq S_i \subseteq U$ , such that for all  $j \in S_i \setminus L$ , we have  $\xi(j, S_i) = \xi^*(j, L, U)$ . (Since  $i \in S_i \setminus L$ , it holds that  $\xi(i, S_i) = \xi^*(i, L, U)$ .)*
- (c) *If for some  $C \subset U$  there is a player  $j \in C$  with  $\xi(j, C) < \xi^*(j, L, U)$  (obviously  $L \not\subseteq C$ ), then there exists at least one set  $T \neq \emptyset$  with  $T \subseteq L \setminus C$ , such that for all  $i \in T$ ,  $\xi(i, C \cup T) = \xi^*(i, L, U)$ .*

An alternative way of expressing conditions (a) and (b) of Fence Monotonicity at some  $L, U$  is the following: We say that a set  $S$  is *optimal* for a player  $i \in S$  if she is serviced with her minimum possible payment in this restriction, i.e.,  $\xi^*(i, L, U) = \xi(i, S)$ . Condition (a) requires the existence of a set that is optimal for all players in it. Condition (b) says that for each agent  $i \in U \setminus L$  belongs to some set  $S_i$  that is optimal for the players in  $S_i \setminus L$ . Notice that for two different players  $i, j$  the sets  $S_i, S_j$  might be different.

Condition (c) is the most important and involved condition. It compares the minimum possible payment of a player in this restriction of the mechanism with her payment when the outcome can be any subset of  $U$ , i.e., it should not necessarily contain the players in  $L$ . Assume that player  $j$  is better off when the outcome is some set  $C$ , where  $L \not\subseteq C$ , i.e., loosely speaking some of the players in  $L \setminus C$  “harm” player  $j$  by their presence in the outcome. Then condition (c) requires the existence of some non-empty set  $T$  with  $T \subseteq L \setminus C$ , such that each player in  $T$  is indifferent between her payment in  $C \cup T$  and her minimum payment in the original restriction.

Observe that if  $\xi$  is cross-monotonic, then the set  $U$  is optimal (and consequently semi-optimal). Hence, conditions (a) and (b) are always satisfied. Moreover, condition (c) is trivially satisfied, as for every  $C \subset U$ , it holds that for all  $i \in C$ ,  $\xi(i, C) \leq \xi(i, U) = \xi^*(i, L, U)$ . As a result, cross-monotonicity implies Fence Monotonicity.

**Proposition 1.** *Let  $\xi$  be a cost sharing scheme*

(i) *If it satisfies condition (a) of Fence Monotonicity for all  $L, U$  with  $|U \setminus L| = 1$ , then it satisfies semi-cross-monotonicity.*

(ii) *If it satisfies conditions (b) of Fence Monotonicity for all  $L, U$  with  $|U \setminus L| = 2$  and (c) of Fence Monotonicity for all  $L, U$  with  $|U \setminus L| = 1$ , then it satisfies the following property: For all  $S \subseteq \mathcal{A}$  and all distinct  $i, j \in S$ , if  $\xi(j, S \setminus \{i\}) < \xi(j, S)$  then  $\xi(i, S \setminus \{j\}) = \xi(i, S)$ .*

Proposition 1 shows exactly how one can derive the two necessary conditions for group-strategyproofness, which were identified by Immorlica et. al. [6]: Part (i) of this Proposition shows how we can derive semi-cross-monotonicity and part (ii) how we can derive the condition identified in Remark B.1 from [6] (Remark B.1 is a special case of part (ii), namely they showed that if  $\xi(i, S \setminus \{j\}) \leq \xi(i, S)$  and  $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$ , then at most one equality can be strict). The two conditions they identified are obtained if apply Fence Monotonicity for cases when  $U \setminus L$  contains either one or two players, i.e., when we restrict the outcome space to only two or four different possible outcomes.

**Theorem 1.** *A cost sharing scheme gives rise to a GSP mechanism if and only if it satisfies Fence Monotonicity.*

### 3.2 Fencing Mechanisms

If you are given a cost sharing scheme that satisfies Fence Monotonicity, it is not straightforward how to construct a GSP mechanism. Fence Monotonicity can yield a GSP mechanism if and only if it is coupled with an allocation rule that satisfies two properties, which we refer to as *Stability* and *Validity* of the tie-breaking rule respectively. We call the underlying mechanisms of this framework Fencing mechanisms.

The mechanisms we design can be put in the following general framework: Given a bid vector as input, we search for a certain pair of sets  $L, U$ , where  $L \subseteq$

$U \subseteq A$  that meets the criteria of Stability we define below and then we choose one of the allocations in this restriction according to a valid tie-breaking rule. If the search for the stable pair is exhaustive, then the resulting algorithm runs in exponential time. Given an arbitrary cost sharing scheme that satisfies Fence Monotonicity we do not know of any polynomial time algorithm for computing the stable pair at every input. However, if we restrict our attention to payments that satisfy stronger conditions like, for example, cross-monotonicity we can come up with a polynomial-time algorithm for finding a stable pair.

**Definition 2 (Stability).** *A pair  $L, U$  is stable at  $b$  if the following conditions are true:*

- For all  $i \in L$ ,  $b_i > \xi^*(i, L, U)$ ,
- for all  $i \in U \setminus L$ ,  $b_i = \xi^*(i, L, U)$ , and
- for all  $R \subseteq A \setminus U$ , there is some  $i \in R$ , such that  $b_i < \xi^*(i, L, U \cup R)$ .

The first Stability condition ensures that each player in  $L$  can be serviced with strictly positive utility. The second Stability condition implies that every player in  $U \setminus L$  can be serviced in at least one outcome but with zero utility. The last property requires that if we try to enlarge  $U$ , then at least one of the newly added players cannot pay in any possible outcome.

After identifying a stable pair these mechanisms output a set  $S$ , where  $L \subseteq S \subseteq U$  given by a tie-breaking function.

**Definition 3 (Validity).** *The mapping  $\sigma : 2^A \times 2^A \times \mathbb{R}^n \rightarrow 2^A$  is a valid tie-breaking rule for  $\xi$ , if for all  $L \subseteq U \subseteq A$ , the set  $S = \sigma(L, U, b)$  satisfies  $L \subseteq S \subseteq U$  and for all  $i \in S$ ,  $\xi(i, S) = \xi^*(i, L, U)$ .*

The dependence on the bid vector allows the mechanism of our framework to change its tie-breaking rule between two bid vectors that share a common stable pair. Obviously, condition (a) of Fence Monotonicity guarantees that a valid tie-breaking rule always exists.

**Definition 4.** *We will say that a mechanism  $(O, \xi)$  is a Fencing Mechanism if and only if*

1.  $\xi$  satisfies Fence Monotonicity and
2. for any bid vector  $b$ ,  $O(b) = \sigma(L, U, b)$ , where  $L, U$  is a stable pair at  $b$  and  $\sigma$  is a valid tie-breaking rule.

It is easy to verify that every Fencing Mechanism satisfies VP from Stability and valid tie-breaking. Moreover, it satisfies CS, because if a player bids higher than any of her payments, then again by Stability she belongs to the set  $L$  and gets serviced.

*Remark 1.* Assume that  $\xi$  is cross-monotonic and let  $S$  be the output of Moulin mechanism for some bid vector  $b$ . Then, the pair  $L, U$ , where  $L = \{i \in S \mid b_i > \xi(i, S)\}$  and  $U = S$ , is the unique stable pair at  $b$ . Moreover, the tie breaking rule  $\sigma(L, U, b) = U$  is always valid. Therefore, Moulin mechanisms can be viewed as a special case of Fencing mechanisms.

**Theorem 2.** *A mechanism is GSP if and only if it is a Fencing Mechanism.*



## 4 Every GSP Mechanism is a Fencing Mechanism

### 4.1 Necessity of Fence Monotonicity

Here, we prove that the cost-sharing scheme of any GSP mechanism satisfies Fence Monotonicity. Let  $(O, \xi)$  be an arbitrary GSP mechanism and consider some  $U \subseteq \mathcal{A}$ . We show that for every  $L \subseteq U$ ,  $\xi$  satisfies each one of the Fence Monotonicity conditions using induction on  $|U \setminus L|$ .

For proving the induction step, we define the notion of a *harm relation* and we prove that it is a strict partial order.

**Lemma 1.** *If  $U' \subseteq U$  and  $L \subseteq L'$ , then for all  $i \in U$ ,  $\xi^*(i, L', U') \geq \xi^*(i, L, U)$ .*

*Proof.* It is obvious that the minimum payment of player  $i$  can only decrease as the set of outcomes, over which the minimum in the definition of  $\xi^*$  is taken, becomes larger.  $\square$

**Definition 5 (Harm relation).** *Fix two sets  $L \subseteq U$  and suppose that  $i, j \in U$ . We say that  $i$  harms  $j$ , if  $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$ . If  $i$  does not harm  $j$ , then  $\xi^*(j, L, U) = \xi^*(j, L \cup \{i\}, U)$  (we get this equality applying Lemma 1).*

**Lemma 2.** *(i) The harm relation satisfies anti-symmetry and transitivity and consequently it is a strict partial order and the induced sub-graph  $G[U \setminus L]$  is a directed acyclic graph.*

*(ii) For every  $i \in L$ , one of the following holds: either every  $j \in U \setminus L$  harms  $i$  or there exists some sink  $k$  of the sub-graph  $G[U \setminus L]$  that does not harm  $i$ .*

*(iii) For every  $i \in U \setminus L$ , one of the following holds: either  $i$  is a sink of the sub-graph  $G[U \setminus L]$  or there exists some a sink  $k$  of the sub-graph  $G[U \setminus L]$  that is harmed by  $i$ .*

*Proof.*

(i) We show this using the induction hypothesis at every  $L \cup \{i\}, U$  for every  $i \in U \setminus L$  and more specifically condition (c) of Fence Monotonicity.

To show anti-symmetry we need to consider only elements of  $U \setminus L$ , as it is impossible that some  $i \in L$  harms any other elements.

Consider two distinct  $i, j \in U \setminus L$  and assume that  $i$  harms  $j$ , that is  $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$ . We will show that  $j$  does not harm  $i$ . From the definition of  $\xi^*$  there is a set  $S_j$ , where  $j \in S_j, L \subseteq S_j \subseteq U$  and  $\xi(j, S_j) = \xi^*(j, L, U)$ . Notice that that  $i \notin S_j$ , since  $\xi^*(j, L \cup \{i\}, U) > \xi(j, S_j)$ . It follows that  $S_j \subset U$  and now we apply the induction hypothesis, and specifically condition (c) of Fence Monotonicity at  $L \cup \{i\}, U$  (the only non-empty subset of  $(L \cup \{i\}) \setminus S_j$  is  $\{i\}$ , so  $T = \{i\}$ ). We get that  $\xi(i, S_j \cup \{i\}) = \xi^*(i, L \cup \{i\}, U)$  or equivalently  $\xi(i, S_j \cup \{i\}) = \xi^*(i, L, U)$ . Since  $L \cup \{j\} \subseteq S_j \cup \{i\} \subseteq U$  we also get that  $\xi^*(i, L \cup \{j\}, U) \leq \xi(i, S_j \cup \{i\})$ . Putting the last two relations together and applying Lemma 1 we get  $\xi^*(i, L \cup \{j\}, U) = \xi^*(i, L, U)$ , which means that  $j$  does not harm  $i$ .

We show transitivity of the harm relation in a similar manner. Consider three distinct players  $i, j$  and  $k$ , where  $i, j \in U \setminus L$  and  $k \in U$  ( $k$  may belong

to  $L$ ) such that  $i$  harms  $j$  and  $j$  harms  $k$ . Assume towards a contradiction that  $i$  does not harm  $k$ . Since  $\xi^*(k, L, U) = \xi^*(k, L \cup \{i\}, U)$ , there is some set  $S_k$ , where  $L \cup \{i\} \subseteq S_k \subseteq U$  such that  $\xi(k, S_k) = \xi^*(k, L, U)$ . Using similar arguments like in the previous part we show that  $\xi(j, S_k \cup \{j\}) = \xi^*(j, L, U)$  and as  $L \cup \{i\} \subseteq S_k \cup \{j\} \subseteq U$  we reach a contradiction from our assumption that  $\xi^*(j, L, U) < \xi^*(j, L \cup \{i\}, U)$ .

(ii) If  $j$  is not a sink of  $G[U \setminus L]$ , there must be an path starting from  $j$ , which ends at a sink  $k$  of this graph. Transitivity implies that  $j$  harms  $k$ .

(iii) Suppose that there is some  $i \in U \setminus L$  that does not harm  $j$ . If  $i$  is a sink of  $G[U \setminus L]$ , setting  $k = i$  completes our proof. Otherwise, there must be path that goes through  $i$  but does not stop there. Let  $k$  be the sink of this path (dag). Notice that transitivity implies that  $i$  harms  $k$ . Thus, it is impossible that  $k$  harms  $j$ , since using transitivity again we would deduce that  $i$  harms  $j$  contradicting our assumption.  $\square$

Lemma 2 allows the effective use of the induction hypothesis in order to identify a successful coalition if some part is violated. We continue by revealing several very interesting and important allocation properties that are satisfied by any GSP mechanism, which we will use to prove the induction step for each property of Fence Monotonicity.

The tables in this section summarize the most important of these allocation properties of GSP mechanisms.

	bids			allocation of a GSP mechanism	
	$L$	$U \setminus L$	$\notin U$		$\forall i \in O(b)$
Lemma 4	$b_i^*$	$\xi^*(i, L, U)$	$-1$	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 7	$> \xi^*(i, L, U)$	$\xi^*(i, L, U)$	$-1$	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 8	$> \xi^*(i, L, U)$	$\in \mathbb{R}$	$-1$	$O(b) \subseteq U$	$\xi(i, O(b)) \geq \xi^*(i, L, U)$

**Table 1.** Allocation properties of GSP mechanisms. Every family of inputs we consider is a subset of the previous one. The last property reveals the following important fact about the allocation of (non-Moulin) GSP mechanisms: If the bids of all players in a set  $L \subseteq U$  have bidden more than their respective minimum payments at  $L, U$ , (and the players in  $A \setminus U$  do not want to participate), then a GSP mechanism never excludes a subset of  $L$  from the outcome in favor of another serviced player, (i.e. so that another player pays less, because the players from  $L$  are not present in the outcome), Loosely speaking the players in  $L$  rule out any outcome  $C \subset U$  such that there is some  $j \in C$  such that  $\xi(j, C) < \xi^*(j, L, U)$ .

**Condition (a) of Fence Monotonicity** For the proof of condition (a) we consider bid vectors where all the players in  $L$  have bidden  $b_i^*$ , all players in  $A \setminus U$  have bidden  $-1$ , and the players in  $U \setminus L$  have bidden exactly  $\xi^*(i, L, U)$ .

We first use the induction hypothesis and the properties of the harm relation to prove that the following Lemma, which is a condition somewhat milder (notice

that here not all players in  $S_j$  but just and one player  $j \in L$  and the players in  $S_j \setminus L$  are guaranteed to pay their minimum payment) than condition (a) of Fence Monotonicity.

**Lemma 3.** *For every  $j \in L$ , there is some set  $S_j$ , where  $L \subseteq S_j \subseteq U$  such that for all  $i \in S_j \setminus L$ ,  $\xi(i, S_j) = \xi^*(i, L, U)$  and  $\xi(j, S_j) = \xi^*(j, L, U)$ .*

*Proof.* Consider some  $j \in L$ . We take two cases as Lemma 2(ii) indicates:

*Case 1:* Suppose that every  $i \in U \setminus L$  harms  $j$ . This implies that  $\xi^*(j, L, U) = \xi(j, L)$  and hence we can set  $S_j = L$ , and Lemma 3 is trivially satisfied, since  $S_j \setminus L = \emptyset$ .

*Case 2:* Consider some sink  $k$  that *does not harm*  $i$ . Applying the induction hypothesis, part (a) of Fence Monotonicity is satisfied at  $L \cup \{k\}, U$ , and there is a set  $S$ , where  $L \cup \{k\} \subseteq S \subseteq U$  such that for all  $i \in S$ ,  $\xi(i, S) = \xi^*(i, L \cup \{k\}, U)$ . Using the fact that  $k$  is a sink and also *does not harm*  $j$  we get that for all  $i \in S' \setminus L$ ,  $\xi^*(i, L \cup \{k\}, U) = \xi^*(i, L, U)$  and  $\xi^*(j, L \cup \{k\}, U) = \xi^*(j, L, U)$ . As a result, we can set  $S_j = S$ .  $\square$

Using this we can prove Lemma 4, which implies that the cost-sharing scheme satisfies condition (a) of Fence Monotonicity.

**Lemma 4.** *At the bid vector  $b$ , where for all  $i \in L$ ,  $b_i = b_i^*$ , for all  $i \in U \setminus L$ ,  $b_i = \xi^*(i, L, U)$ , and for all  $i \notin U$ ,  $b_i = -1$ , it holds that  $L \subseteq O(b) \subseteq U$  and for all  $i \in O(b)$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ .*

*Setting  $S = O(b)$ , condition (a) of Fence Monotonicity is satisfied at  $L, U$ .*

*Proof.* By CS and VP the output of the mechanism satisfies  $L \subseteq O(b) \subseteq U$ . First note that in any case a player  $i \in U \setminus L$  has utility zero: either she is not serviced, or by VP, if she is serviced her payment cannot exceed her bid and cannot be less than her minimum payment  $\xi^*(i, L, U)$  so  $\xi(i, O(b)) = b_i = \xi^*(i, L, U)$ . What remains is to show that every  $j \in L$  also pays  $\xi^*(j, L, U)$ .

Suppose towards a contradiction that for some player  $j \in L$ ,  $\xi(j, O(b)) > \xi^*(j, L, U)$  (If the set  $U \setminus L$  is empty  $U = L$  it is impossible that some  $j \in L$  is charged more than  $\xi^*(j, L, U) = \xi(j, L)$ ). Then she could form a coalition with the players in  $U \setminus L$ , who would enforce the set  $S_j$ , where  $S_j$  is the set guaranteed to exist by Lemma 3, to be output i.e. the players  $i \in U \setminus L$  could change their bids to  $b_i^*$  if  $i \in S_j$  and  $-1$  if  $i \notin S_j$ , so that by CS and VP the output is  $S_j$ .

Since for every  $i \in S_j \setminus L$ ,  $\xi(i, S_j) = \xi^*(i, L, U)$  their utilities remain zero after this manipulation, while the utility of player  $j$  strictly increases, and thus the coalition is successful indeed.

Consequently for all  $j \in L$ ,  $j \in O(b)$  and  $\xi(j, O(b)) = \xi^*(j, L, U)$  and the players in  $O(b) \setminus L$  are charged at their minimum payment as well.  $\square$

**Condition (b) of Fence Monotonicity** Using Lemma 2(ii), we can split the proof in two cases: First we consider the sinks, as the fulfillment of the second condition of Fence Monotonicity for a sink is an immediate consequence of the induction hypothesis.

**Lemma 5.** *For every sink  $k$  of  $G[U \setminus L]$  condition (b) of Fence Monotonicity is satisfied at  $L, U$ .*

*Proof.* Using the induction hypothesis at  $L \cup \{k\}, U$  we get from part (a) of Fence Monotonicity that there is a set  $S$ , where  $L \cup \{k\} \subseteq S \subseteq U$ , such that for all  $i \in S$ ,  $\xi(i, S) = \xi^*(i, L \cup \{k\}, U)$ . Since  $k$  is a sink, she does not harm any other player from  $U \setminus L$ , and so we have that for all  $i \in S \setminus L$ ,  $\xi^*(i, L, U) = \xi^*(i, L \cup \{k\}, U)$ . Thus, setting  $S_k = S$  condition (b) of Fence Monotonicity is satisfied for  $k$  at  $L, U$ .  $\square$

We already know that Fence Monotonicity is satisfied for for sinks and the induction hypothesis also guarantees that it is satisfied for  $L \cup \{i\}, U$ , i.e. if we add to  $L$  some player  $i \notin L$ . To use this fact in order to prove Fence Monotonicity for all other cases as well, we prove that if Fence Monotonicity (b) is satisfied just for one player  $j$  at some  $L', U'$ , then if the input is  $b$  (as defined in the next lemma), this player receives service and is charged  $\xi^*(j, L', U')$ .

**Lemma 6.** *Consider some  $L' \subseteq U' \subseteq \mathcal{A}$ . Suppose that the set  $S_j$  as in condition (b) of Fence Monotonicity exists just for some player  $j \in U' \setminus L'$ . Then at any bid vector  $b$  with  $b_i = b_i^*$  for  $i \in L'$ ,  $b_j > \xi^*(j, L', U')$ ,  $b_i = \xi^*(i, L', U')$  for all  $i \in U' \setminus (L' \cup \{j\})$ , and  $b_i = -1$  for all  $i \notin U'$ , it holds that  $j \in O(b^j)$  and  $\xi(j, O(b)) = \xi^*(j, L', U')$ .*

*Proof.* Like in the proof of Lemma 4 we show that By CS and VP the output of the mechanism satisfies  $L \subseteq O(b) \subseteq U$ . Moreover, all players in  $U \setminus (L \cup \{j\})$  have utility zero. We need to show that  $j$  is serviced and charged  $\xi^*(j, L, U)$ .

Suppose towards a contradiction that player  $j$ , is either not serviced or is charged  $\xi(j, O(b)) > \xi^*(j, L, U)$ . Obviously if she is serviced at a higher payment then she prefers the set  $S$  to the current outcome. If she is not serviced she again prefers  $S$ , as we assumed that  $b_j > \xi^*(j, L, U) = \xi(j, S)$  and thus her utility would become positive if the outcome was  $S$ . In a similar manner like in the proof of Lemma 4 she could form a coalition, with the players in  $U \setminus L$ , enforcing the output  $S$ . Again our assumption about the payments of the rest of the agents in  $S \setminus L$  implies that their utility remains unchanged thus this coalition is successful.  $\square$

Now consider a player  $j$  in  $U \setminus L$ , that is *not a sink* of  $G[U \setminus L]$  and let  $k$  be one of its *sinks* such that  $j$  *harms*  $k$ . In order to prove that the second condition is satisfied for  $j$ , we will invoke group-strategyproofness at certain bid vectors (trying to generalize Example 2 where  $j$  takes the role of player 3 and  $k$  the role of player 4).

Let  $\epsilon > 0$  be an arbitrarily small quantity. We gradually reason about the allocation of the bid vectors  $b^k, b^{j,k}$  and finally  $b^j$ , which are defined in the following table.

bids	$i \in L$	$j$	sink $k$	$i \in U \setminus L$	$i \in \mathcal{A} \setminus U$
$b^k$	$b_i^*$	$\xi^*(j, L, U)$	$\xi^*(k, L, U) + \epsilon$	$\xi^*(i, L, U)$	-1
$b^{j,k}$	$b_i^*$	$\xi^*(j, L, U) + \epsilon$	$b_k^*$	$\xi^*(i, L, U)$	-1
$b^j$	$b_i^*$	$\xi^*(j, L, U) + \epsilon$	$\xi^*(k, L, U) + \epsilon$	$\xi^*(i, L, U)$	-1

The allocation properties that we show for the bids vectors we just defined are summarized in the following Table.

	output	payment of $j$	payment of the sink $k$
Claim 1	$b^k$	$j \notin O(b^k)$	$\xi(k, O(b^k)) = \xi^*(k, L, U)$
Claim 2	$b^{j,k}$	$\xi(j, O(b^{j,k})) = \xi^*(j, L, U)$	$\xi(k, O(b^{j,k})) > \xi^*(k, L, U)$
Claim 3	$b^j$	$\xi(j, O(b^j)) = \xi^*(j, L, U)$	$k \notin O(b^j)$

*Claim 1.* At the bid vector  $b^k$  the following hold:

- (a) player  $k$  is serviced and charged  $\xi^*(k, L, U)$ ,
- (b) player  $j$  is not serviced.

*Claim 2.* At the bid vector  $b^{j,k}$  the following hold

- (a) For all  $i \in U \setminus (L \cup \{j, k\})$ ,  $b_i^{j,k} = \xi^*(i, L \cup \{k\}, U)$  and  $b_j^{j,k} = \xi^*(j, L \cup \{k\}, U) + \epsilon$ .
- (b) Player  $j$  is serviced and charged  $\xi^*(j, L, U)$ .
- (c) Player  $k$  is serviced and charged more than  $\xi^*(k, L, U)$ .

*Claim 3.* At the bid vector  $b^j$  the following hold

- (a) Player  $k$  is not serviced at  $b^j$ .
- (b) Player  $j$  is serviced at  $b^j$ .
- (c)  $L \subseteq O(b^j) \subseteq U$  and every player  $i \in O(b^j) \setminus L$ , is charged  $\xi^*(i, L, U)$ .

**Corollary 1.** *Condition (b) of Fence Monotonicity is also satisfied for any non-sink  $j$  at  $L, U$  by setting  $S_j = O(b^j)$ . (We already know it is satisfied for sinks by Lemma 5).*

**Condition (c) of Fence Monotonicity** To show that the cost-sharing scheme satisfies the third property of Fence Monotonicity, at  $L, U$ , we need the induction hypothesis only for showing (as we have already done) that condition (a) of Fence Monotonicity is satisfied at this pair and specifically only the allocation properties of Lemma 4. Now we consider inputs for which we do not get anymore directly from CS, that the players in  $L$  surely receive service. The idea is to gradually generalize the bid vectors and characterize the allocation of the mechanism. The following table contains allocation properties that any GSP mechanism satisfies.

**Lemma 7.** *For every bid vector  $b$ , where for all  $i \in L$ ,  $b_i > \xi(i, S)$ , for all  $i \in U \setminus L$ ,  $b_i = \xi^*(i, L, U)$  and for all  $i \notin U$ ,  $b_i = -1$ , it holds that  $L \subseteq O(b) \subseteq U$  and for all  $i \in O(b)$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ .*

*Proof.* We will prove our statement by the induction on the cardinality of the set  $T = \{i \in L \mid b_i \neq b_i^*\}$ .

**Base:** Since  $T = \emptyset$ , we simply apply Lemma 4.

**Induction step:** For the induction step we will show that  $L \subseteq O(b) \subseteq U$  and for all  $i \in L$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ . Then it is easy to see that every  $i \in O(b) \setminus L$  must be serviced at  $\xi^*(i, L, U)$  by VP and the definition of  $\xi^*$ .

By the construction of the bid vector we have that  $O(b) \subseteq U$ . Consider now some  $j \in T$ . The induction hypothesis implies that  $j$  is serviced and charged  $\xi^*(j, L, U)$  at the bid vector  $(b_j^*, b_{-j})$ . Suppose that  $j$  is not serviced at  $b$  resulting in zero utility. Since  $b_j - \xi^*(j, L, U) > 0$  she can misreport  $b_j^*$  so as to increase her utility to a strictly positive quantity and thus violate strategyproofness of the mechanism. As a result,  $j$  must be serviced at  $b$  and hence by strategyproofness  $j$  must be serviced at the same payment. (since  $j$  is an arbitrary element of  $T$ , the same holds for all  $j \in T$ ).

Now since  $j$  is indifferent between the two outcomes group-strategyproofness requires the same about the rest of the players. This is only possible if every  $i \in L \setminus T$  ( $i \in O(b)$  by CS since  $b_i = b_i^*$ ) is charged the same payment in either input i.e.  $\xi(i, O(b)) = \xi(i, O(b_j^*, b_{-j})) = \xi^*(i, L, U)$  (induction hypothesis).  $\square$

	bids			allocation of a GSP mechanism	
	$L$	$U \setminus L$	$\notin U$		$\forall i \in O(b)$
Lemma 4	$b_i^*$	$\xi^*(i, L, U)$	-1	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 7	$> \xi^*(i, L, U)$	$\xi^*(i, L, U)$	-1	$L \subseteq O(b) \subseteq U$	$\xi(i, O(b)) = \xi^*(i, L, U)$
Lemma 8	$> \xi^*(i, L, U)$	$\in \mathbb{R}$	-1	$O(b) \subseteq U$	$\xi(i, O(b)) \geq \xi^*(i, L, U)$

**Table 2.** Allocation properties of GSP mechanisms. Every family of inputs we consider is a subset of the previous one. Last property sheds light on an important property of the allocation of (non-Moulin) GSP mechanisms. It reveals the following fact: If the bids of all players in a set  $L \subseteq U$  have surpassed their respective minimum payments at  $L, U$ , (and the players in  $\mathcal{A} \setminus U$  do not want to participate), then a GSP mechanism never excludes a subset of  $L$  from the outcome in favor of another serviced player, who would pay less if they were not present in the outcome, Loosely speaking the players in  $L$  rule out any outcome  $C \subset U$  such that there is some  $j \in C$  such that  $\xi(j, C) < \xi^*(j, L, U)$ .

**Lemma 8.** *For every bid vector  $b$ , where for all  $i \in L$ ,  $b_i > \xi(i, S)$ , for all  $i \in U \setminus L$ ,  $b_i \in \mathbb{R}$  and for all  $i \notin U$ ,  $b_i = -1$ , it holds that: for all  $i \in O(b)$ ,  $\xi(i, O(b)) \geq \xi^*(i, L, U)$ .*

*Proof.* Let  $b^0$  be any bid vector satisfying the conditions of Lemma 7, which means that for all  $i \in L$ ,  $b_i^0 > \xi^*(i, L, U)$ , for all  $i \in U \setminus L$ ,  $b_i^0 = \xi^*(i, L, U)$  and for all  $i \notin U$ ,  $b_i^0 = -1$ . Then we relax the constraints we put on the bids of the players in  $U \setminus L$  (the rest of the players bid always according to  $b^0$ ) and prove that no player becomes serviced at a price strictly less than her minimum payment at  $L, U$ .

Consider some bid vector  $b$ . Like in the proof of Lemma 7 let  $T = \{i \in U \setminus L \mid b_i \neq b_i^0\}$ . However, we will need some new definitions here. Let

$$R := \{i \in T \mid i \in O(b) \text{ and } \xi(i, O(b)) > b_i^0\} \quad (1)$$

When the true values are given by  $b^0$  and all the players announce  $b$  instead, the set  $R$  represents the set of players who participate in the coalition and *lose utility* (it was zero and becomes strictly negative). Finally, let  $m$  represent the number of relations of the form  $\xi^*(i, L, U) \leq \xi(i, S) < b_i$  for all  $i \in R$ , i.e.,

$$m = |\{(i, S) \mid i \in S \text{ and } i \in T \text{ and } \xi^*(i, L, U) \leq \xi(i, S) < b_i\}| \quad (2)$$

Notice that a set  $S$  may contribute more than once to this number and that if  $R \neq \emptyset$  then  $m > 0$ , since for all  $i \in T$ , there is some  $S$  such that  $\xi(i, S) = \xi^*(i, L, U) = b_i^0 < \xi(i, O(b)) \leq b_i$ , where the last inequality follows from VP.

We are now ready to state our induction argument. We show that the allocation of the mechanism satisfies for all  $i \in O(b)$ ,  $\xi(i, O(b)) \geq \xi^*(i, L, U)$  for any bid vector  $b$ , where for all  $i \notin U \setminus L$ ,  $b_i^0 = b_i$  using induction on  $|T| + m$ .

**Base:** For  $|T| + m = 0$  we have that  $b = b^0$  and the allocation property follows from Lemma 7.

**Induction step:** Assume that there is some  $j \in O(b)$  that is charged less than  $\xi^*(j, L, U)$ . Since by VP  $O(b) \subseteq U$ , either  $j \in L$  which implies that  $b_j^0 > \xi^*(j, L, U)$  or  $j \in U \setminus L$  and thus  $b_j^0 = \xi^*(j, L, U)$ . In both cases we have that

$$j \in O(b) \text{ and } \xi(j, O(b)) < \xi^*(j, L, U) \leq b_j^0. \quad (3)$$

We will prove that there exists *at least one* successful coalition, contradicting the assumed group-strategyproofness of the mechanism. Note that  $j \notin R$ .

*Case 1:* If  $R \subset T$ , we construct the bid vector  $b'$ , where for every  $i \in R$ ,  $b'_i = b_i$  and for every  $i \notin R$ ,  $b'_i = b_i^0$ . We complete the proof of this case by showing that  $(T \setminus R) \cup \{j\}$  form a coalition when their true values are  $b'$  bidding  $b$ .

First, we prove that in the truthful scenario every player  $i \in T \setminus R$  has zero utility and non-negative after the misreporting (profile  $b$ ). By using the induction hypothesis at  $b'$  (differs from  $b^0$  in  $|R| < |T|$  coordinates) we get that for every  $i \in T \setminus R$  such that  $i \in O(b')$  (the rest of the players have obviously zero utility), it holds that  $\xi^*(i, L, U) \leq \xi(i, O(b')) \leq b'_i$ , where the last inequality follows from VP. Since  $i \in T \setminus R$ , we get that  $b'_i = b_i^0 = \xi^*(i, L, U)$  and thus  $b'_i = \xi(i, O(b'))$ . Now consider the outcome after the misreporting. Either  $i \notin O(b)$  and hence she has zero utility after the misreporting or  $i \in O(b)$  and her utility becomes  $b'_i - \xi(i, O(b)) = b_i^0 - \xi(i, O(b)) \geq 0$ , where the last inequality follows from Equation 1 as  $i \in T \setminus R$ .

Second, we prove that player  $j$  strictly increases her utility. Considering the truthful scenario there are two cases for  $j$ : She is serviced at  $b'$  and charged  $\xi(j, O(b')) \geq \xi^*(j, L, U)$  from the induction hypothesis. From Equation 3 we get that  $j$  is serviced after the misreporting and charged  $\xi(j, O(b)) < \xi^*(j, L, U)$ , and thus her payment decreases, while she is still serviced. Now if  $j$  is not serviced at  $b'$ , then she has zero utility. From Equation 3 we get that  $b_j^0 > \xi(j, O(b))$  and since  $j \notin R$  it follows that  $b_j^0 = b'_j$ . As a result, her utility increases to  $b'_j - \xi(j, O(b)) > 0$ .

*Case 2:* Otherwise if  $R = T$ , then first we show that it must be the case that for all  $i \in R$ ,  $b_i = \xi(i, O(b))$ . Assume that the converse holds, i.e., for some  $i \in R$ ,  $b_i > \xi(i, O(b))$ . Consider the bid vector  $(\xi(i, O(b)), b_{-i})$ . If she is serviced then by strategyproofness her payment should be the same. Notice that by lowering the bid of player  $i$ , it holds that the quantity  $|T| + m$  is reduced by at least one, since she does not count the relation  $b_i > \xi(i, O(b)) > \xi^*(i, L, U)$  anymore. Thus, by the induction hypothesis player  $j$  is either not serviced or she is charged an amount greater than or equal to  $\xi^*(i, L, U)$ . When the true values are given by  $(\xi(i, O(b)), b_{-i})$ , players  $i$  and  $j$  can form a successful coalition bidding  $b$ : The utility of player  $i$  is zero in either scenario, while the utility of player  $j$  increases since either her payment increases or if she was not serviced in the truthful scenario her utility becomes  $b_j - \xi(j, O(b)) > 0$ .

As a result, it must be the case that for all  $i \in R$ ,  $b_i = \xi(i, O(b))$ . Notice that by VP we have that  $O(b) \subseteq U$  (for every  $i \in \mathcal{A} \setminus U$ , it holds that  $i \notin R$  and thus  $b_i = b_i^0 = -1$ ). Since  $\xi^*(j, L, U) > \xi(j, O(b))$  there must be that  $L \not\subseteq O(b)$ . We show that when the true values are given by  $b$  the players in  $L \setminus O(b)$  and in  $T$  can form a successful coalition bidding  $b^0$ .

Assume that the true values are given by  $b$ . The utility of every  $i \in T$  is zero since they are charged an amount equal to their bids. Obviously, the same holds for all  $k \in L \setminus O(b)$ .

First, we prove that the utility of every  $i \in T$  remains non-negative after the misreporting. Consider some  $i \in T$  such that  $i \in O(b^0)$  (obviously the rest players have zero utility). From Lemma 7 it holds that  $\xi(i, O(b^0)) = \xi^*(i, L, U)$  and since  $\xi^*(i, L, U) = b_i^0$  ( $i \in U \setminus L$ ) and  $b_i^0 < \xi(i, O(b)) = b_i$ , we conclude that her utility increases to  $b_i - \xi(i, O(b^0)) > 0$ .

Second, we show that the utility of every  $k \in L \setminus O(b)$  increases, as it becomes strictly positive. Since  $k \notin T$  we get that  $b_k = b_k^0$ . Moreover, since  $k \in L$  we have  $b_k^0 > \xi^*(k, L, U)$  and from Lemma 7 we get that  $k \in O(b^0)$  and  $\xi(k, O(b^0)) = \xi^*(k, L, U)$ . As a result, her utility becomes  $b_k - \xi(k, O(b^0)) > 0$ .  $\square$

Using Lemma 8, it is easy to show that any GSP mechanism satisfies condition (c) of Fence Monotonicity.

*Claim 4.* Consider any  $C \subset U$  such that there is some  $j \in C$  with  $\xi(j, C) < \xi^*(j, L, U)$ . At the bid vector  $b^c$ , where for all  $i \in C$ ,  $b_i^c = b_i^*$ , for all  $i \in L \setminus C$ ,  $b_i^c = \xi^*(i, L, U) + \epsilon$ , and for all  $i \notin C \cup L$ ,  $b_i^c = -1$ , it holds that

- (a) for all  $i \in O(b^c) \setminus C$ ,  $\xi(i, O(b^c)) = \xi^*(i, L, U)$ ,
- (b)  $C \subset O(b^c) \subseteq L \cup C$  and
- (c) for all  $i \in O(b^c) \setminus C$ , it holds that  $\xi(i, O(b^c)) = \xi^*(i, L, U)$

Setting  $T = O(b^c) \setminus C$  we can conclude that every GSP mechanism satisfies condition (c) of Fence Monotonicity.

## 4.2 Necessity of Stability and Valid tie-breaking.

Next, we show that the allocation of every GSP mechanism satisfies Stability and uses a Valid tie-breaking rule. Since its cost-sharing scheme must satisfy Fence Monotonicity, we can prove the following generalization of Lemma 7.



**Lemma 9.** *Let  $L \subseteq U \subseteq \mathcal{A}$ . For every bid vector  $b$  such that  $L, U$  is stable, it holds that  $L \subseteq O(b) \subseteq U$  and for all  $i \in O(b)$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ .*

*Proof.* Notice that the only difference between the bid vectors we consider here and the ones we considered in Lemma 7 is that there every  $i \in \mathcal{A} \setminus U$  could only bid  $-1$ . Our proof here uses again a simmilar induction on  $|T|$ , where  $T = |\{i \in L \mid b_i \neq b_i^*\}|$ . We show that  $L \subseteq O(b)$  and for all  $i \in O(b)$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ .

**Base:** First, by CS we have that  $L \subseteq U$ . Next, we show that no player from  $\mathcal{A} \setminus U$  is serviced. Suppose towards a contradiction that  $O(b) \setminus U \neq \emptyset$ , and let  $R := O(b) \setminus U$  the subset of players of  $\mathcal{A} \setminus U$  who receive service. We will show that VP is violated. By the definition of the bid vectors we consider there is some  $i \in R$ , such that  $b_i < \xi^*(i, L, U \cup R) \leq \xi(i, O(b))$  (since  $L \subseteq O(b) \subseteq U \cup R$ ) and thus we reach a contradiction.

As a result, it holds that  $L \subseteq O(b) \subseteq U$ . Thus, for every  $i \in O(b)$ , we have that  $\xi(i, O(b)) \geq \xi^*(i, L, U)$ . By VP we get the equality for every  $i \in O(b) \setminus L$ . Now assume that the previous inequality is strict for some  $j \in L$ . We show that the players in  $(\mathcal{A} \setminus U) \cup \{j\}$  can form a successful coalition, where every  $i \notin U$  announces  $-1$ . The utilities of these players remain zero after the misreporting and applying Lemma 7 we get that  $j$  is serviced and charged  $\xi^*(j, L, U) < \xi(j, O(b))$ .

**Induction Step:** We show that  $L \subseteq O(b)$  and for all  $i \in L$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$  exactly like in the proof of Lemma 7. Now since  $L \subseteq O(b)$  in a similar way we show that  $U \subseteq O(b)$ . Thus, this restriction together with the definition of the bid vector, implies that for every  $i \in O(b) \setminus L$ ,  $\xi(i, O(b)) = \xi^*(i, L, U)$ .  $\square$

Lemma 9 implies that an allocation should satisfy Stability and Validity of the tie-breaking rule, whenever a stable pair exists. Lemma 15 (which is proven in the next section) guarantees the existence of a unique stable pair at every input, which completes our proof.

## 5 The classes of GSP and Fencing Mechanisms coincide

We complete our characterization by proving that Fencing Mechanisms are GSP. Throughout this section we fix a cost-sharing scheme  $\xi$  that satisfies Fence Monotonicity, and a valid tie-breaking rule  $\xi$  for  $\sigma$ . First, we provide some technical Lemmas regarding Stability and Fence Monotonicity.

**Lemma 10.** *For every bid vector  $b$  and set  $L$  with  $L \subseteq \mathcal{A}$ , there exists a unique set  $U$  with  $U \supseteq L$ , such that for all  $i \in U \setminus L$  we have  $b_i \geq \xi^*(i, L, U)$  and any other set with the same property is a subset of  $U$ . Moreover, the pair  $L, U$  satisfies condition 3. of Stability.*

*Proof.* Assume towards a contradiction that there exist two distinct sets  $U_1, U_2$  none of which is a subset of the other that both satisfy this property. Then we could construct the set  $U_1 \cup U_2$  which also satisfies the same property and is

a proper superset of both  $U_1$  and  $U_2$  reaching a contradiction. Indeed for all  $i \in U_1 \setminus L$  we have  $b_i \geq \xi^*(i, L, U_1) \geq \xi^*(i, L, U_1 \cup U_2)$  as the minimum payment of player  $i$  can only decrease as the set of outcomes, over which the minimum in the definition of  $\xi^*$  is taken, becomes larger. A similar inequality holds for the players in  $i \in U_2 \setminus L$  completing the proof.

Now we show that if  $U$  is the maximal set with this property, then  $L, U$  satisfy property 3. of stability. Consider some non-empty  $R \subseteq \mathcal{A} \setminus U$ . Assume that for all  $i \in R$ ,  $b_i \geq \xi^*(i, L, U \cup R)$ . From Lemma 1 we have that for all  $i \in U \setminus L$ ,  $b_i \geq \xi^*(i, L, U \cup R)$  and thus we reach contradiction by the maximality of  $U$ .  $\square$

**Lemma 11.** *Suppose that  $L, U$  is a stable pair at the bid vector  $b$  and that  $S$  is set with the property that for all  $i \in S$  we have that  $b_i - \xi(i, S) \geq 0$ . (If the mechanism would output  $S$  then all players are would have been served with non-negative utility.) If (a)  $S \not\subseteq U$ , or*

*(b) if  $S \subseteq U$  and for some  $i \in S$  we have  $\xi^*(i, L, U) > \xi(i, S)$ , there exists some non-empty set  $T \subseteq L \setminus S$ , such that for all  $j \in T$  we have  $\xi(j, S \cup T) < b_j$ .*

*Proof.* We will first show that if  $S \not\subseteq U$  then there exists some  $i \in S$  we have  $\xi^*(i, L, U \cup S) > \xi(i, S)$ . Since  $L, U$  is stable (property 3.) there exists some  $i \in S \setminus U$  such that  $b_i < \xi^*(i, L, U \cup S)$  and since from the initial assumption  $\xi(i, S) \leq b_i$  we get that  $\xi(i, S) < \xi^*(i, L, U \cup S)$

The rest of the proof is the same for both cases (note that in what follows, if  $S \subseteq U$  then  $S \cup U = U$ ). Note that  $S \subset U \cup S$ , since  $L \not\subseteq S$ . Applying part (c) of Fence Monotonicity we get that there exists a non-empty  $T \subseteq L \setminus S$  such that for all  $j \in T$  we have  $\xi(j, S \cup T) = \xi^*(j, L, U \cup S) \leq \xi^*(j, L, U)$ , where the last inequality is by the definition of  $\xi^*$ , since the minimum cannot decrease as the set of outcomes over which it is taken becomes larger. From property 1. of stability and since  $T \subseteq L$ , for all  $j \in T$  we have  $\xi^*(j, L, U) < b_j$ . Consequently for all  $j \in T$  we have  $\xi(j, S \cup T) < b_j$ .  $\square$

**Lemma 12.** *Suppose that  $L \subseteq S \subseteq U$  and that there exists a non-empty  $T \subseteq \mathcal{A} \setminus S$  such that for all  $i \in T$  we have  $b_i \geq \xi(i, S \cup T)$  and that for at least one player from  $T$  the inequality is strict. Then  $L, U$  is not a stable set at the bid vector  $b$ .*

*Proof.* Suppose towards a contradiction that  $L, U$  is stable at  $b$  and that there exists some  $T \subseteq \mathcal{A} \setminus S$ , such that for all  $i \in T$ ,  $b_i \geq \xi(i, S \cup T)$  and that for at least one player the inequality is strict.

Since  $L \subseteq S \cup T \subseteq U \cup T$  from the definition of  $\xi^*$  we get that for all  $i \in T$  we have  $\xi^*(i, L, U \cup T) \leq \xi^*(i, L, U) \leq \xi(i, S \cup T)$ .

If  $T \subseteq U$ , then there exists some  $i \in T$  such that  $b_i > \xi(i, S \cup T) \geq \xi^*(i, L, U)$ . Since  $T \subseteq \mathcal{A} \setminus L$  this contradicts with stability (condition 2.) of  $b$  at  $L, U$ .

If  $T \not\subseteq U$ , then  $T \setminus U$  is not empty and for all  $i \in T \setminus U$  we have  $b_i \geq \xi(i, S \cup T) \geq \xi^*(i, L, U \cup T)$ , which contradicts with stability (condition 3.) of  $b$  at  $L, U$ .  $\square$

## 5.1 Uniqueness and Group-strategyproofness

**Lemma 13.** *For the inputs where there exists a stable pair it, it is unique.*

*Proof.* Assume that there are two stable pairs  $L_1, U_1$  and  $L_2, U_2$ . Suppose first that  $L_1 \neq L_2$  and without loss of generality that  $L_2 \not\subseteq L_1$ . Consequently there exists some  $j \in L_2 \setminus L_1$ .

By Fence Monotonicity (part (a)) there exists a set  $S_2$  such that  $L_2 \subseteq S_2 \subseteq U_2$  and  $\xi(i, S_2) = \xi^*(i, L_2, U_2)$  for all  $i \in S_2$ . Notice that for all  $j \in S_2$ ,  $b_j \geq \xi(j, S_2)$ , where strict inequality holds only if  $j \in L_2$ . The idea is to apply Lemma 11 and get that there exists a non-empty  $T \subseteq L \setminus S_2$ , such that for all  $i \in T$  we have  $b_i > \xi(i, S_2 \cup T)$ . We will then apply Lemma 12 to show that  $L_2, U_2$  is not stable at  $b$  which contradicts our initial assumption.

It only remains to show that we can apply Lemma 11. If  $S_2 \not\subseteq U_1$  this is immediate. Suppose that  $S_2 \subseteq U_1$ . If  $j \in L_2 \setminus L_1$  then also  $j \in U_1 \setminus L_1$  thus from stability (condition 2.) of  $L_1, U_1$ , we get that  $b_j = \xi^*(j, L_1, U_1)$  and from stability (condition 1.) of  $L_2, U_2$  and since  $j \in L_2$ , we get that  $b_j > \xi(j, S_2)$ . Therefore, we get that  $\xi(j, S_2) < \xi^*(j, L_1, U_1)$  and consequently  $S_2$  satisfies the requirements of Lemma 11.

We showed that  $L_1 = L_2 = L$ . Suppose towards a contradiction that  $U_1 \neq U_2$ . From Lemma 10 there exists a unique maximal set  $U$  such that  $b_i \geq \xi^*(i, L, U)$  for all  $i \in U$ . Consequently  $U_1, U_2$  are subsets of  $U$  and at least one of them, say  $U_1$  a proper subset of  $U$ . Then the players in  $U \setminus U_1$  contradict stability (condition 3.).  $\square$

**Lemma 14.** *Fencing mechanisms are GSP at inputs where a stable pair exists.*

With this we mean that given two bid vectors where there exists a stable pair, then the allocation of a Fencing mechanism satisfies group-strategyproofness locally, i.e., if we just restrict the true values to be either of these vectors and have the agents misreport the other.

*Proof.* Let  $b$  and  $b'$  be two bid vectors, and let  $L, U$  and  $L', U'$  be their corresponding unique (from Lemma 13) stable pairs and  $O(b)$  and  $O(b')$  the corresponding outputs. Assume towards a contradiction that some of the players can form a successful coalition when the true values are  $b$  reporting  $b'$ .

We will first show that any player  $i$  served in the new outcome,  $i \in O(b')$ , has non-negative utility i.e.  $b_i \geq \xi(i, O(b'))$ . Take some  $i$  that is output in  $O(b')$ . If  $b_i = b'_i$  then it holds trivially since the mechanism satisfies VP at the outcome  $O(b')$ . If  $b_i \neq b'_i$  then  $i$  changes his bid to be part of the coalition and consequently his utility after this coalition is non-negative, which gives  $b_i - \xi(i, O(b')) \geq 0$ .

The next step is to apply Lemma 11 to show that there exists some non-empty  $T \subseteq L \setminus O(b')$  such that for all  $i \in T$  we have  $b_i > \xi(i, O(b') \cup T)$ . If  $O(b') \not\subseteq U$  then the premises of the Lemma hold trivially.

Suppose that  $O(b) \subseteq U$ . For the coalition to be successful the utility of at least one player  $j$  increases strictly when the players bid  $b'$  consequently  $j \in O(b')$  and  $b_j - \xi(j, O(b')) > 0$ . We will show that  $\xi^*(j, L, U) > \xi(j, O(b'))$ .

If  $j$  is not served at  $b$  and since  $O(b') \subseteq U$  from stability we get that  $j \in U \setminus L$  and  $b_j = \xi^*(j, L, U) > \xi(j, O(b'))$ . If  $j$  is served at  $b$  then her payment equals  $\xi^*(j, L, U)$  by the definition of the mechanism and in order that she profits strictly it must be  $\xi(j, O(b')) < \xi^*(j, L, U)$ , so we can again apply Lemma 11.

Finally we will show that for all  $i \in T$  we have  $b_i = b'_i$ . After the manipulation the players in  $T$  are not serviced, while as  $T \subseteq L$  from stability we have that in the truthful scenario the players in  $T$  are serviced with positive utility. Consequently the players in  $T$  wouldn't have an incentive to be part of the coalition and change their bids.

Putting everything together we get that there exists a  $T \subseteq \mathcal{A} \setminus O(b')$  such that for all  $i \in T$  we have  $b'_i > \xi(i, O(b') \cup T)$ , which by Lemma 12 contradicts our initial assumption that  $L', U'$  is stable at  $b'$ .  $\square$

## 5.2 Existence of Stable Pairs

The next Lemma completes our statement that Fencing Mechanisms are GSP.

**Lemma 15.** *For every bid vector  $b$  there exists a unique stable pair. Consequently, Fencing Mechanisms are meaningful and GSP.*

*Proof.* Let  $b_i^*$  be some value that is big enough so that player  $i$  always gets serviced, you can let  $b_i^* > \max_{S \subseteq \mathcal{A}} \xi(i, S)$ , since the allocation of the mechanism satisfies Stability. We will show that there exists a stable pair at any input  $b$  by induction on the number  $m$  of coordinates that are less than  $b_i^*$ , i.e. on the number  $m = |\{i \mid b_i < b_i^*\}|$ .

**Base:** For  $m = 0$ , we only have to show that there exists a stable pair for the bid vector  $(b_1^*, \dots, b_n^*)$  and  $\mathcal{A}, \mathcal{A}$  is a stable pair.

**Induction Step:** Suppose that if a bid vector has  $m - 1$  coordinates that are less than  $b_i^*$ , then it has a stable pair. We will show that if a bid vector  $b$  has  $m$  coordinates that are less than  $b_i^*$  then it also has a stable pair. We will suppose towards a contradiction that there exists no stable pair at  $b$ .

We first need some definitions. Let  $L^* := \{i \mid b_i \geq b_i^*\}$  and  $U^*$  the corresponding (from Lemma 10) maximal set. Notice that for all  $i \in L^*$ ,  $b_i = b_i^* > \xi^*(i, L^*, U^*)$  by the definition of  $b_i^*$ , which implies that the pair  $L^*, U^*$  satisfies the first condition of stability. Moreover, from Lemma 6 we get that the third condition of stability is satisfied for  $L^*, U^*$  as well. Thus, as  $L^*, U^*$  cannot be stable at  $b$  (from our assumption), the set  $W = \{i \mid i \in U^* \setminus L^* \text{ and } b_i > \xi^*(i, L^*, U^*)\}$  should be non-empty.

For each  $i \in W$  we define a corresponding pair  $L_i, U_i$  as follows: The pair  $L_i, U_i$  is the unique (by Lemma 13) stable pair of  $(b_i^*, b_{-i})$ , which exists by the induction hypothesis.

*Claim 5.* If  $L_i, U_i$  is the stable pair of  $(b_i^*, b_{-i})$ , then

- (a)  $L^* \cup \{i\} \subseteq L_i$  and (b)  $U_i \subseteq U^*$ .
- (c) If  $j \in L_i \setminus (L^* \cup \{i\})$  then  $j \in W$  and  $b_j > \xi^*(L^*, U^*)$ .

*Claim 6.* If there exists no stable pair at  $b$ , then for all  $i \in W$  we have

- (a)  $b_i \leq \xi^*(i, L_i, U_i)$ , and (b)  $L^* \cup \{i\} \subset L_i$ .

Since  $W \neq \emptyset$  there exists some  $i \in W$  such that  $L_i$  has minimum cardinality, i.e.  $i = \arg \min_{i \in W} |L_i|$ . By Claim 6 (b) there exists some  $j \in L_i \setminus (L^* \cup \{i\})$  and by Claim 5 (c)  $j \in W$ . We will show that  $L_j \subset L_i$  contradicting the choice of  $i$ .

*Claim 7.* If there exists no stable pair at  $b$  and  $j \in L_i \setminus (L^* \cup \{i\})$  then

- (a) The pair  $L_i, U_i$  is stable at the bid vector  $(b_i^*, b_j^*, b_{-\{i,j\}})$ .
- (b)  $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$  and  $i \notin L_j$ .
- (c) If there exists some  $k \in L_j \setminus L_i$ , then the mechanism is not GSP at inputs where a stable pair exists.

*Proof.* (a) The pair  $L_i, U_i$  is stable at bid vector  $(b_i^*, b_j^*, b_{-\{i,j\}})$ , since  $L_i, U_i$  is stable at  $(b_i, b_{-i})$  and since raising the bid of player  $j$  ( $j \in L_i$  from our initial assumption) to  $b_j^*$  does not effect its stability.

(b) From stability of  $L_i, U_i$  at  $(b_i^*, b_{-i})$  we get that  $b_j > \xi^*(j, L_i, U_i)$ , while from part (a) of Claim 6 we get that  $\xi^*(j, L_j, U_j) \geq b_j$ . Consequently  $\xi^*(j, L_j, U_j) > \xi^*(j, L_i, U_i)$ .

Supposing towards a contradiction that  $i \in L_j$  we also get in a similar way as before that the pair  $L_j, U_j$  is stable at  $(b_i^*, b_j^*, b_{-\{i,j\}})$ . By Lemma 13 these two pairs coincide, which is a contradiction because we just showed that the payment of  $j$  is different.

(c) We will show that if there exists some  $k \in L_j \setminus L_i$  then the mechanism is not GSP at inputs, where a stable pair exists. Observe that  $k \neq i, j$ .

Consider first the vector  $b^{i,j} := (b_i^*, b_j^*, b_{-\{i,j\}})$ , which stable pair is  $L_i, U_i$  from part (a). As  $k \notin L_i$  either  $k$  is not serviced, or if  $k$  is serviced, then her utility is zero. As for  $j$  she is serviced with payment  $\xi^*(j, L_i, U_i) < \xi^*(j, L_j, U_j) = \xi(j, O(b_j^*, b_{-j}))$  (from (a)).

Consider then  $b^j := (b_j^*, b_{-j})$ , where  $L_j, U_j$  is stable. As  $k \in L_j$  and  $k$  is serviced and  $b_k > \xi(k, O(b_j^*, b_{-j}))$ .

Resuming player  $j$  strictly prefers  $O(b^{i,j})$  to  $O(b^j)$ , while for  $k$  the situation is exactly the opposite. The idea is to construct a bid vector  $b'$  where  $i$  has zero utility and either  $\{i, j\}$  or  $\{i, k\}$  is a successful coalition. Let  $b' = (\xi(i, O(b^{i,j})), b_j^*, b_{-\{i,j\}})$  and notice that induction hypothesis implies that there is a stable pair at  $b'$ . Moreover, observe that the three bid vectors differ only on the bid of player  $i$  and consequently from strategyproofness, at every input  $i$  is served, she is charged  $\xi(i, O(b^{i,j}))$ . This implies that if  $i$  is served at  $b'$  she is charged an amount equal to her bid. Moreover,  $i$  is serviced at  $b^j$  in the degenerate case where  $b' = b^j$  as otherwise  $VP$  would imply that  $\xi(j, O(b^j)) \leq b_j^j < b_j^{i,j} = \xi(j, O(b^{i,j}))$ . In every case we have that  $i$  has zero utility in at  $b'$  and  $b^j$ .

Observe first that  $k$  must be serviced at  $b'$  and charged  $\xi(k, O(b')) \leq \xi(k, O(b^j))$  as otherwise  $\{i, k\}$  would have been able to form a successful coalition when the true values are  $b'$  bidding  $b^j$ . Similarly we can show that  $\xi(j, O(b')) \leq \xi(j, O(b^{i,j}))$ , excluding the degenerate case  $b' = b^j$ , because otherwise  $\{i, j\}$  would have been able to form a successful coalition when the true values are  $b'$  bidding  $b^{i,j}$  (the utility of  $i$  is kept to zero as in the truthful scenario).

As a result players  $j$  and  $k$  strictly prefer  $O(b')$  to  $O(b^j)$  and  $O(b^{i,j})$  respectively. If  $i \in O(b')$  then  $\{i, k\}$  when true utilities are given by  $b^{i,j}$  can form a successful coalition bidding  $b'$  strictly increasing the utility of  $k$ , while keeping the utility of  $i$  constant, since she is still served with the same payment. If  $i \notin O(b')$  then we deduce that  $\{i, j\}$  when the true utilities are given by  $b^j$  can form successful coalition bidding  $b'$  strictly increasing the utility of  $j$ , while keeping the utility of  $i$  to zero.  $\square$

Now since  $j \in W$  and from Claim 7  $i \notin L_j$  and for every  $k \in L_j$  we have that also  $k \in L_i$ , we get that  $L_j \subset L_i$ , which completes the proof.  $\square$

## 6 Budget Balance and Complexity

We demonstrate the use of our characterization by showing that even in the case of three players GSP mechanisms fail in having a constant budget balance.

**Theorem 3.** *There are cost function families where GSP mechanism has arbitrarily poor budget balance ratio.*

*Proof.* Let  $\mathcal{A} = \{1, 2, 3\}$ . Consider the cost sharing function defined on  $\mathcal{A}$  as follows:  $C(\{1, 2\}) = C(\{1, 3\}) = 1$ ,  $C(\{1\}) = C(\{2\}) = C(\{3\}) = x$ ,  $C(\{2, 3\}) = x^2 + x$  and  $C(\{1, 2, 3\}) = x^3 + x^2 + x$ , where  $x \geq 1$ . We show that there is no  $\frac{1}{x}$ -budget balanced cost sharing scheme, that satisfies Fence Monotonicity.

Assume by contradiction that there is a  $\alpha$ -budget balanced cost sharing scheme that satisfies Fence Monotonicity for  $C$  where  $1 \geq \alpha > \frac{1}{x}$ .

First, we consider payments of the players 2 and 3 at the sets  $\{1, 2\}$  and  $\{1, 3\}$  respectively. From the upper bound of budget balance we have that  $\xi(2, \{1, 2\}) \leq 1$  and  $\xi(3, \{1, 3\}) \leq 1$ . From Proposition 1 (b) it cannot be the case that both of these players payment increases at the set  $\{1, 2, 3\}$ . That is either  $\xi(2, \{1, 2, 3\}) \leq 1$  or  $\xi(3, \{1, 2, 3\}) \leq 1$  (or both). W.l.o.g. we assume that  $\xi(2, \{1, 2, 3\}) \leq 1$ . We consider now two cases for  $\xi(2, \{2, 3\})$ .

*Case 1:* Suppose that  $\xi(2, \{2, 3\}) \leq 1$ . From the lower bound of budget-balance at the set  $\{2\}$ , we have that  $\xi(2, \{2\}) > 1$ , and thus  $\xi(2, \{2, 3\}) < \xi(2, \{2\})$ . We apply Proposition 1 (b) and deduce that  $\xi(3, \{2, 3\}) \leq \xi(3, \{3\}) \leq C(3) = x$ . Thus, the total sum of the shares at the set  $\{2, 3\}$  is  $\xi(2, \{2, 3\}) + \xi(3, \{2, 3\}) \leq x + 1$ , which contradicts our assumption about the budget balance of the mechanism, since  $C(\{2, 3\}) = x^2 + x$ .

*Case 2:* Suppose that  $\xi(2, \{2, 3\}) > 1 \geq \xi(2, \{1, 2, 3\})$ . Notice that the contra-positive implication of Proposition 1 (a) implies that  $\xi(3, \{1, 2, 3\}) \leq \xi(3, \{2, 3\}) \leq C(\{2, 3\}) - \xi(2, \{2, 3\}) < x^2 + x - 1$ . Moreover, as the payment of player 2 decreases from  $\{2, 3\}$  to  $\{1, 2, 3\}$ , Proposition 1 (b) implies that the payment of player 1 doesn't increase from  $\{1, 3\}$  to  $\{1, 2, 3\}$ , i.e.  $\xi(1, \{1, 2, 3\}) \leq \xi(1, \{1, 3\}) \leq C(\{1, 3\}) = 1$ . Now summing the shares at the set  $\{1, 2, 3\}$ , we get that  $\xi(1, \{1, 2, 3\}) + \xi(2, \{1, 2, 3\}) + \xi(3, \{1, 2, 3\}) \leq x^2 + x + 1$ , which is again contradicts our budget balance assumption, since  $C(\{1, 2, 3\}) = x^3 + x^2 + x$ .  $\square$

Notice that Fencing Mechanism run in exponential time, as the only way we know to compute its outcome is to search exhaustively all possible pairs and check if they are stable. In fact, we only need to search the lower set  $L$ , since there is a polynomial time algorithm (which is defined in the proof of the next lemma) for checking if there is a set  $U$  such that  $L, U$  is stable.

A natural question that arises in the is if it is computationally more efficient to find the appropriate outcome than identifying the stable pair.

**Theorem 4.** *Suppose that we are given the outcome of a GSP mechanism at  $b$ . Given polynomial access to  $\xi^*(i, L, U)$  for all  $L \subseteq U \subseteq \mathcal{A}$  and all  $i \in U$ , we can identify the stable pair in polynomial time.*

*Proof.* Consider the following process, which takes as input a bid vector  $b$  and a set  $L$ .

```

repeat
   $U \leftarrow L \cup \{i \in U \setminus L \mid b_i \geq \xi^*(i, L, U)\}$ 
until For all  $i \in U \setminus L$ ,  $b_i \geq \xi^*(i, L, U)$ 

```

We will prove that if we feed this process with the lower set  $L$  of the stable pair at  $b$ , then the outcome is the upper set of the stable pair.

Obviously by the definition of the process, for its final set  $U$  it holds that for all  $i \in U$ ,  $b_i \geq \xi^*(i, L, U)$ . First, we show that  $U$  is the maximal set with this property.

Assume that there is some  $U'$  with  $U' \not\subseteq U$ , that satisfies this property, then we have that for all  $i \in (U \cup U') \setminus L$ ,  $b_i \geq \xi^*(i, L, U \cup U')$ . Thus, it is impossible that the players in  $U \cup U'$  are removed at any step of the previous process, contradicting our assumption that  $U \neq U' \cup U$  is the outcome of this process.

Now consider now the upper set  $U''$  of the stable pair at  $b$ . Since  $U''$  satisfies this property it follows that  $U'' \subseteq U$ . Notice that if  $U'' \subset U \neq \emptyset$ , then the elements of  $U \setminus U''$  would violate stability of  $L, U''$ . Thus,  $U'' = U$  and consequently this process outputs the upper set of the stable pair.

As a result, given an outcome  $S$  of a GSP mechanism we can compute  $L := \{i \in S \mid b_i > \xi(i, S)\}$  and use this process to find the upper set  $U$ . The time-complexity of this algorithm is polynomial in the number of players assuming that we have polynomial-time access to every  $\xi^*(i, L, U)$  for all  $L \subseteq U \subseteq \mathcal{A}$  and all  $i \in U$ .  $\square$

## 7 Conclusion and future directions

We believe that the most interesting future directions are the following: How can our characterization be applied for obtaining cost-sharing mechanisms with better budget-balance guarantees or lower bounds for specific problems?

Does there exist a polynomial-time algorithm for finding the allocation of a cost-sharing scheme that satisfies Fence Monotonicity or maybe can we show that the problem of finding a stable pair is computationally hard? Given such a result, how should our characterization be restricted when we add tractability as an additional requirement?

Finally, our characterization could be used as a guidance for characterizing mechanisms for specific cost-sharing problems which can more effectively capture the hardness of the problem.

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## A Examples of Mechanisms that violate just one part of Fence Monotonicity and are not GSP

We will give three representative examples to illustrate, why a cost sharing scheme, which does not satisfy Fence Monotonicity, cannot give rise to a group-strategyproof mechanism. We chose our examples in a way that only one condition of Fence Monotonicity is violated and only at a specific pair  $L, U$ . (in fact in condition (c), the violation is present at two pairs, however it can be shown that this is unavoidable.)

*Example 1 (a).*

Let  $\mathcal{A} = \{1, 2, 3, 4\}$ . We construct a cost sharing scheme, such that condition (a) of Fence Monotonicity is not satisfied at  $L = \{1, 2\}$  and  $U = \{1, 2, 3, 4\}$ , as follows.

$\xi$	1	2	3	4	$\xi$	1	2	3	4	$\xi$	1	2	3	4
$\{1, 2, 3, 4\}$	30	30	30	30	$\{1, 2\}$	30	30	–	–	$\{3, 4\}$	–	–	30	30
$\{1, 2, 3\}$	20	30	30	–	$\{1, 3\}$	20	–	30	–	$\{1\}$	30	–	–	–
$\{1, 2, 4\}$	30	20	–	30	$\{1, 4\}$	30	30	–	–	$\{2\}$	–	30	–	–
$\{1, 3, 4\}$	30	–	20	30	$\{2, 3\}$	–	30	30	–	$\{3\}$	–	–	30	–
$\{2, 3, 4\}$	–	30	30	20	$\{2, 4\}$	–	20	–	30	$\{4\}$	–	–	–	30

Consider the bid vector  $b := (b^*_1, b^*_2, 30, 30)$ . Notice that players 3 and 4 are indifferent to being serviced or not, as the single value they may be charged as payment equals their bid. Moreover, notice that either player 1 or player 2 (or both) must pay 30 strictly over their minimum payment 20 under this restriction. Without loss of generality assume that  $\xi(1, O(b)) = 30$ . Consider the bid vector  $b' := (b^*_1, b^*_2, b^*_3, -1)$ . By VP and CS it holds that  $O(b') = \{1, 2, 3\}$  and thus  $\xi(1, O(b')) < \xi(1, O(b))$ . Notice that the utilities of players 3 and 4 remain zero, and consequently  $\{1, 3, 4\}$  form a successful coalition. In a similar manner we prove the existence of successful coalition when  $\xi(2, O(b)) = 30$ .

*Example 2 (b).*



Let  $\mathcal{A} = \{1, 2, 3, 4\}$ . We construct a cost sharing scheme, such that condition (b) of Fence Monotonicity is not satisfied at  $L = \{1, 2\}$  and  $U = \{1, 2, 3, 4\}$  for player 3, as follows.

$\xi$	1	2	3	4	$\xi$	1	2	3	4	$\xi$	1	2	3	4
$\{1, 2, 3, 4\}$	30	30	30	30	$\{1, 2\}$	30	30	-	-	$\{3, 4\}$	-	-	30	30
$\{1, 2, 3\}$	30	30	40	-	$\{1, 3\}$	30	-	30	-	$\{1\}$	30	-	-	-
$\{1, 2, 4\}$	30	30	-	20	$\{1, 4\}$	30	-	-	30	$\{2\}$	-	30	-	-
$\{1, 3, 4\}$	30	-	30	30	$\{2, 3\}$	-	30	30	-	$\{3\}$	-	-	30	-
$\{2, 3, 4\}$	-	30	30	30	$\{2, 4\}$	-	30	-	30	$\{4\}$	-	-	-	30

Consider the bid vector  $b^{3,4} := (b_1^*, b_2^*, 35, b_4^*)$ . Strategyproofness implies that  $3 \in O(b^{3,4})$ , since otherwise if she is not serviced (zero utility), she can misreport  $b_3^*$  changing the outcome to  $\{1, 2, 3, 4\}$  and increasing her utility to  $35 - 30 > 0$ .

Next, consider the bid vector  $b^4 := (b_1^*, b_2^*, 30, 25)$ . Assume that  $4 \notin O(b^4)$ . Moreover, notice that by VP it is impossible that  $3 \in O(b^4)$ . Thus,  $\{3, 4\}$  can form a successful coalition bidding  $b' = (b_1^*, b_2^*, -1, b_4^*)$ , changing the outcome to  $\{1, 2, 4\}$  increasing the utility of player 4 to  $25 - 20 > 0$ , while keeping the utility of player 3 at zero.

Finally, consider the bid vector  $b^3 := (b_1^*, b_2^*, 35, 25)$ . Notice that the  $b^3$  differs with  $b^4$  and  $b^{3,4}$  in the coordinates that correspond to players 3 and 4 respectively. Like in the case of  $b^4$  the only possible outcomes by VP and CS at  $b^3$  are  $\{1, 2, 4\}$  and  $\{1, 2\}$ .

Assume that player 4 is serviced at  $b^3$ , which implies that  $\xi(4, O(b^4)) < \xi(4, O(b^{3,4}))$ . This contradicts strategyproofness, since when the true values are  $b^{3,4}$ , player 4 can bid according to  $b^3$  in order to decrease her payment and still being serviced.

Now, assume that player 4 is not serviced at  $b^3$ . Then  $\{3, 4\}$  can form a successful coalition when true values are  $b^3$  bidding  $b^4$ , increasing the utility of player 4 to  $25 - 20 > 0$ , while keeping the utility of player 3 at zero.

*Example 3 (c).*

Let  $\mathcal{A} = \{1, 2, 3, 4\}$ . This time we construct a cost sharing scheme, such that part (c) is not satisfied for  $L = \{1, 2\}$  (or  $\{1, 2, 3\}$ ) and  $U = \{1, 2, 3, 4\}$  and specifically for  $C_i = \{3, 4\}$  and  $i = 3$ .

$\xi$	1	2	3	4	$\xi$	1	2	3	4	$\xi$	1	2	3	4
$\{1, 2, 3, 4\}$	30	30	30	30	$\{1, 2\}$	20	20	-	-	$\{3, 4\}$	-	-	20	30
$\{1, 2, 3\}$	20	20	30	-	$\{1, 3\}$	20	-	20	-	$\{1\}$	30	-	-	-
$\{1, 2, 4\}$	30	30	-	30	$\{1, 4\}$	30	30	-	-	$\{2\}$	-	30	-	-
$\{1, 3, 4\}$	30	-	30	30	$\{2, 3\}$	-	20	20	-	$\{3\}$	-	-	30	-
$\{2, 3, 4\}$	-	30	30	30	$\{2, 4\}$	-	30	-	30	$\{4\}$	-	-	-	30

Suppose that the values are  $b := (25, 25, b_3^*, 30)$ . The feasible by VP outcomes are  $\{1, 2, 3\}$ ,

$\{1, 3\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$  and  $\{3\}$ . Notice that player 4 has zero utility regardless of the outcome.

Assume that  $O(b) \neq \{1, 2, 3\}$  and w.l.o.g that  $1 \notin O(b)$ . Then  $\{1, 2, 4\}$  can form a successful coalition bidding  $b' := (b_1^*, b_2^*, b_3^*, -1)$  increasing the utility of player 1 to  $25 > 0$  without decreasing the utility of player 2 (either  $O(b) = \{2, 3\}$  and  $\xi(2, O(b')) = \xi(2, O(b))$  and her utility remains the same utility or  $2 \notin O(b)$  and her utility increases to  $25 > 20$ , like in the case of player 1) and keeping the utility of player 4 at zero.

Finally, assume that  $O(b) = \{1, 2, 3\}$ . Then  $\{3, 4\}$  can form a successful coalition bidding  $b'' := (25, 25, b_3^*, b_4^*)$ . Obviously  $\{3, 4\} \subseteq O(b'')$ . Notice that VP excludes each of the following outcomes:  $\{1, 3, 4\}$ ,  $\{2, 3, 4\}$  and  $\{1, 2, 3, 4\}$ . As a result,  $O(b'') = \{3, 4\}$  implying that the utility of player 3 increases, as  $\xi(3, O(b)) > \xi(3, O(b''))$ , while player 4 keeps her utility at zero.

## B Missing proofs

### B.1 Condition (b) of Fence Monotonicity

**Proof of Claim 1** We get that player  $k$  is serviced and charged  $\xi^*(k, L, U)$  by applying Lemma 6 for  $k$  with  $L' = L$  and  $U' = U$ .

(b) Suppose towards a contradiction that  $j \in O(b^k)$ . The payment of player  $k$  would be lower bounded by  $\xi^*(k, L \cup \{j\}, U)$ , since  $L \cup \{j\} \subseteq O(b^k) \subseteq U$ , which contradicts with the fact that  $k$  is charged  $\xi^*(k, L, U)$ , which is strictly lower by our assumption that  $j$  harms  $k$ .  $\square$

**Proof of Claim 2** (a) Since  $k$  is a sink of  $G[U \setminus L]$ , we have that for all  $i \in U \setminus (L \cup \{j, k\})$ ,  $\xi^*(i, L \cup \{k\}, U) = \xi^*(i, L, U) = b_i^{j,k}$ . Similarly, we get that  $\xi^*(j, L \cup \{k\}, U) + \epsilon = \xi^*(j, L, U) + \epsilon = b_j^{j,k}$ .

(b) We apply Lemma 6 for  $j$  with  $L' = L \cup \{k\}$  and  $U' = U$ , since condition (b) is satisfied for  $j$  at this pair (induction hypothesis) and the bid vector satisfies the requirements of this Lemma. ( $b_i^{j,k} = b_i^*$  for all  $i \in L \cup \{k\}$ ,  $b_i^{j,k} = -1$  for all  $i \notin U$  and using Claim 2 (a).) As a result,  $j \in O(b^{j,k})$  and  $\xi(j, O(b^{j,k})) = \xi^*(j, L \cup \{k\}, U) = \xi^*(j, L, U)$ .

(c) From part (b) we get that the set  $O(b^{j,k})$  satisfies that  $L \cup \{j\} \subseteq O(b^{j,k}) \subseteq U$ , and thus the payment of player  $k$  is lower bounded by  $\xi^*(k, L \cup \{j\}, U)$ . Since  $j$  harms  $k$  we get that  $\xi^*(k, L \cup \{j\}, U) > \xi^*(k, L, U)$  completing our proof.  $\square$

**Proof of Claim 3** (a) Assume that player  $k$  is serviced at  $b^j$ . Notice that by VP and the definition of  $\epsilon$ , if  $k$  is serviced at  $b^j$  then her payment cannot exceed  $\xi^*(k, L, U)$ . Additionally, notice that the only coordinate  $b^j$  differs from  $b^{j,k}$  is the bid of player  $k$ . Thus, strategyproofness is violated by  $b^{j,k}$ , since from Claim 2 the payment of  $k$  decreases.

(b) Suppose that  $j$  is not serviced at  $b^j$ . Then  $\{j, k\}$  can form a successful when true values are  $b^k$  bidding  $b^j$ , since from Claim 1 the utility of  $k$  increases

from zero to  $\epsilon$  and the utility of  $j$  is kept to zero (she is not serviced at either input).

(c) By VP and CS we get that  $L \subseteq O(b) \subseteq U$ , thus the payment of every serviced player  $i$  is lower bounded by  $\xi^*(i, L, U)$ . By definition of  $b^j$  and VP of the mechanism we get the equality for every player in  $O(b) \setminus (L \cup \{j\})$ . Moreover, from the definition of  $\epsilon$ , we conclude the same for  $j$ .  $\square$

## B.2 Condition (c) of Fence Monotonicity

**Proof of Claim 4** (a) Immediate by applying Lemma 8.

(b) VP and CS imply that  $C_j \subseteq O(b^c) \subseteq L \cup C_j$ . The strict inclusion  $C_j \subset O(b^c)$  follows from (a) and the initial assumption of the Lemma that  $\xi(j, C_j) < \xi^*(j, L, U)$ .

(c) From (a) we have that for all  $i \in O(b^c) \setminus C_j$ ,  $\xi(i, O(b^c)) \geq \xi^*(i, L, U)$ . However all of these players have bidden  $\xi^*(i, L, U) + \epsilon$ , where  $\epsilon$  is an arbitrarily small positive number, smaller than any payment difference, and by VP would not receive service if they had to pay higher than their bid. So  $\xi(i, O(b^c)) = \xi^*(i, L, U)$ .  $\square$

## B.3 Existence of Stable Pairs

**Proof of Claim 5** (a) Using the definition of  $L^*$  we get that  $L^* \cup \{i\}$  contains all the players who have bidden higher than any payment of the mechanism in  $(b_i^*, b_{-i})$ . Since  $L_i$  is stable at  $(b_i^*, b_{-i})$ , each one of these players, who have bidden strictly higher than any payment of the mechanism, should be serviced (any value higher than any payment satisfies the definition of CS for Fencing Mechanisms), and if they are serviced they obviously have strictly positive utility, and thus  $L^* \cup \{i\} \subseteq L_i$ .

(b) The idea is to show that for all  $j \in (U^* \cup U_i) \setminus L^*$ ,  $b_j \geq \xi^*(j, L^*, U^* \cup U_i)$  and since we defined  $U^*$  be the maximal set with this property, we get that  $U_i \subseteq U^*$ .

From definition of  $U^*$  we have that for all  $j \in U^* \setminus L^*$ ,  $b_j \geq \xi^*(j, L^*, U^*) \geq \xi^*(j, L^*, U^* \cup U_i)$ , where the last inequality follows from Lemma 1.

Now consider some  $j \in U_i \setminus U^*$  ( $j \neq i$ ). Since  $L_i, U_i$  is stable at  $(b_i^*, b_{-i})$  we get that  $b_j \geq \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U_i \cup U^*)$  by applying Lemma 1, since from (a)  $L^* \subset L_i$ .

(c) From (a),(b) and Lemma 1 we have that  $\xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$ . As we defined  $L_i, U_i$  to be the stable pair at  $(b_i^*, b_{-i})$  and  $j \in L_i$ , for  $j \neq i$  we have  $b_j > \xi^*(j, L_i, U_i) \geq \xi^*(j, L^*, U^*)$  and as  $j \notin L^*$ , we have  $j \in W$ .  $\square$

**Proof of Claim 6** (a) The pair  $L_i, U_i$  is stable at  $(b_i^*, b_{-i})$  (by definition of  $L_i, U_i$ ) but it is not stable at  $b$  (from our assumption that there exists no stable pair at  $b$ ). Since the two bid vectors differ only on the  $i$ -th coordinate we deduce that stability (condition 1. as  $i \in L_i$ ) is not satisfied by the  $i$ -th coordinate of  $b$ , thus  $b_i \leq \xi^*(i, L_i, U_i)$ .

(b) From Claim 5 (a) we already have that  $L_i \supseteq L^* \cup \{i\}$ . Suppose towards a contradiction that  $L_i = L^* \cup \{i\}$ . From Fence Monotonicity (condition (b)) we get that there exists some  $S_i$ , where  $L^* \subseteq S_i \subseteq U^*$  and  $i \in S_i$  such that for all  $j \in S_i \setminus L^*$  we have  $\xi(j, S_i) = \xi^*(j, L_i, U_i)$ .

The idea is to show that  $L_i \subseteq S_i \subseteq U_i$ , which implies that  $\xi^*(i, L_i, U_i) \leq \xi(i, S_i) = \xi^*(i, L^*, U^*)$ . Considering also that  $\xi^*(i, L^*, U^*) < b_i$ , because  $i \in W$ , we get then that  $\xi^*(i, L_i, U_i) < b_i$ , contradicting the inequality we showed in (a).

It only remains to show that  $L_i \subseteq S_i \subseteq U_i$ . Notice first that  $L_i \subseteq S_i$  since  $L_i = L^* \cup \{i\}$ . Moreover, since for all  $j \in S_i \setminus L_i$ , it holds that  $b_j \geq \xi(j, S_i)$  and the bids of these players are the same at  $(b_i^*, b_{-i})$ , stability of  $L_i, U_i$  (condition 3) implies that  $S_i \subseteq U_i$ , because otherwise  $S_i \setminus U_i \neq \emptyset$  and its elements would violate condition 3. of stability.  $\square$

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