Optimal Auctions vs. Anonymous Pricing

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Abstract

For selling a single item to agents with independent but non-identically distributed values, the revenue optimal auction is complex. With respect to it, Hartline and Roughgarden [1] showed that the approximation factor of the second-price auction with an anonymous reserve is between two and four. We consider the more demanding problem of approximating the revenue of the ex ante relaxation of the auction problem by posting an anonymous price (while supplies last) and prove that their worst-case ratio is e. As a corollary, the upper-bound of anonymous pricing or anonymous reserves versus the optimal auction improves from four to e. We conclude that, up to an e factor, discrimination and simultaneity are unimportant for driving revenue in single-item auctions.

1. Introduction

Methods from theoretical computer science are amplifying the understanding of studied phenomena broadly. A quintessential example from auction theory is the following. Myerson [2] is oft quoted as showing that the second-price auction with a reserve price is revenue optimal among all mechanisms for selling a single item. This result is touted as triumph for microeconomic theory as in practice reserve-pricing mechanisms are widely prevalent, e.g., eBay's auction. A key assumption in this result, though, is that the agents in the auction are a priori identical; moreover, relaxation of this assumption renders the theoretically optimal auction much more complex and infrequently observed in practice. Optimality of reserve pricing with agent symmetry, thus, does not explain its prevalence broadly in asymmetric settings, e.g., eBay's auction where agents can be distinguished by public bidding history and reputation. This paper considers the approximate optimality of anonymous pricing and auctions with anonymous reserves, e.g., eBay's buy-it-now pricing and auction, and justifies their wide prevalence in asymmetric environments.

With two agents, anonymous reserve pricing is a tight two approximation to the optimal auction; moreover, a surprising corollary of a main result of Hartline and Roughgarden [1] showed that the second-price auction with anonymous reserves is generally no worse than a four approximation. The question of resolving the approximation factor within $[2, 4]$ has remained open for the last half decade. Technically, (a) tight methods for understanding symmetric solutions in asymmetric environments are undeveloped, and (b) the main method for analyzing auction revenue is by Myerson's virtual values but for this question virtual values give a mixed sign objective that renders challenging the analysis of approximation. The four approximation of Hartline and Roughgarden [1] employs the only known approach for resolving (b), an approximate extension of the main theorem of Bulow and Klemperer [3]. Our approach directly takes on the challenge of (a) by giving a tight analysis of anonymous pricing versus a standard upper bound (described below); corollaries of this analysis are tightened upper bounds on approximation of the optimal auction from four to $e \approx 2.718$ for both anonymous pricing and anonymous reserves.

In the Bayesian single-item auction problem agents' values are drawn from a product distribution and expected revenue with respect to the distribution is to be optimized. Our development of the approximation bound for anonymous pricing and reserves is based on the analysis of four classes of mechanisms:

1) **Ex ante relaxation (a discriminatory pricing):** An ex ante pricing relaxes the feasibility constraint of the auction problem, from selling at most one item ex post, to selling at most one item in expectation over the draws of agents' values, i.e., ex ante. Fixing a probability of serving a given agent the optimal ex ante mechanism offers this agent a posted price irrespective of the outcome of the mechanism for the other agents. This relaxation was identified as a quantity of interest in Chawla et al. [4] and its study was refined by Alaei [5] and Yan [6].

2) **Auction:** An auction is any mechanism that maps values to outcome and payments subject to incentive and feasibility constraints. The optimal auction was characterized by Myerson [2] and this characterization, though complex, is the foundation of modern auction theory.

3) **Anonymous reserve:** An anonymous reserve mechanisms is a variant of the second-price auction where bids below an anonymous reserve are discarded, the winner is the highest of the remaining agents, and the price charged is the maximum of the remaining agents' bids or the reserve if none other remain.

4) **Anonymous pricing**: An anonymous pricing mechanism posts an anonymous price and the first agent to arrive who is willing to pay this price will buy the item.

For any distribution over agents' values the optimal revenue attainable by each of these classes of mechanisms is non-increasing with respect to the above ordering. The final inequality of optimal anonymous reserve exceeding optimal anonymous pricing follows as with equal reserve and price, the former has only higher revenue as competition drives a higher price. The ex ante relaxation is a quantity for analysis only, while the other problems yield relevant mechanisms.

Our main technical theorem identifies the supremum over all instances of the ratio of the revenues of the optimal ex ante relaxation to the optimal anonymous pricing as the solution to an equation which evaluates to e. To our knowledge, this evaluation is not by any standard progressions or limits that where previously known to evaluate to e. The theorem assumes that the distribution of agents' values satisfies a standard *regularity* property that is satisfied by common distributions, e.g., uniform, normal, exponential; without this assumption we show that the approximation factor is n for n-agent environments (see Section 5).

Theorem 1.1 (Anonymous pricing versus ex ante relaxation). *For a single item environment with agents with independently (but non-identically) distributed values from regular distributions, the worst case approximation factor of anonymous pricing to the ex ante relaxation is* $\left(\mathcal{V}\left(\mathcal{Q}^{-1}(1)\right) + 1\right)$ *which evaluates to* $e \approx 2.718$ *where*

$$\mathcal{V}(p) \triangleq p \cdot \ln\left(\frac{p^2}{p^2 - 1}\right), \qquad\qquad \mathcal{Q}(p) \triangleq \int_p^\infty -\frac{\mathcal{V}'(v)}{v} \, \mathrm{d}v.$$

Intuition for the theorem, as given by functions $\mathcal{V}(\cdot)$ and $\mathcal{Q}(\cdot)$, and its proof is as follows. We write a mathematical program to maximize the worst case approximation factor; a tight-in-the-limit continuous relaxation of this program gives the objective $1 + \mathcal{V}(p)$ subject to $\mathcal{Q}(p) \leq 1$ which has the following interpretation. There is a continuum of agents and each agent value distribution is given by a pointmass at a value with some probability (and then a continuous distribution below the pointmass to minimally satisfy the regularity property). The function $\mathcal{V}(p)$ is the expected pointmass value from agents with pointmass value at least price p;[1] $\mathcal{Q}(p)$ is the expected number of these agents to realize their pointmass value. The optimal p^* meets the constraint with equality, i.e., $\mathcal{Q}(p^*) = 1$.

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[1] $\mathcal{V}(\cdot)$ excludes the contribution from the "highest valued agent" which is 1; hence the objective $1 + \mathcal{V}(p)$.
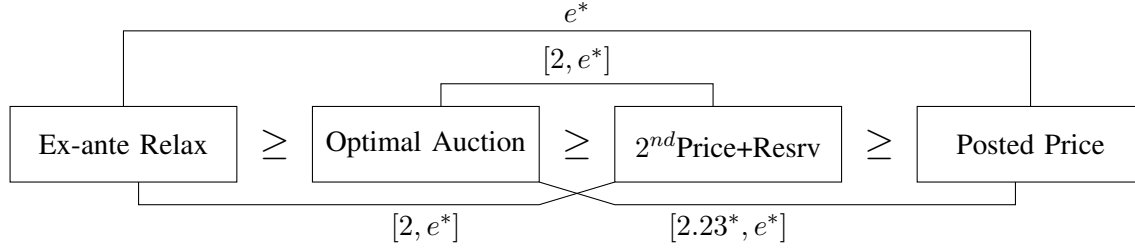
Figure 1. Revenue gap between mechanisms of study; * denotes new bounds.

Corollaries of this theorem are the improved upper bounds by e (from 4) on the worst-case approximation factor of anonymous reserves and anonymous pricing with respect to the optimal auction. On the worst case instance of the theorem, however, the actual approximation factor of anonymous reserve and anonymous pricing are 2 and 2.23, respectively (see Section 6). The latter improves on the known lower bound of two; the former does not improve the known lower bound. The question of refining our understanding of the revenue of anonymous reserves on worst-case instances and identifying a tight bound with respect to the optimal auction remains open. See Figure 1.

The corollary relating anonymous pricing to the optimal auction has implications on mechanism design for agents with multi-dimensional preferences (e.g., for multiple items, see Chawla et al. [4]). Understanding these problems, though there has been considerable recent progress, remains an area with fundamental open questions for optimization and approximation. Recently, Haghpanah and Hartline [7] proved the optimality of uniform pricing for a single unit-demand buyer with values drawn from a large family of item-symmetric distributions. An immediate corollary of our anonymous pricing result is that, for a unit-demand buyer with values drawn from an asymmetric product distribution, uniform pricing is an e approximation (improved from four) to the optimal non-uniform pricing (cf. Cai and Daskalakis [8]) and, via a result of Chawla et al. [9], a $2e$ approximation to the optimal pricing over lotteries (i.e., randomized allocations, improved from eight). Further refinement of this latter bound remains an important open question. These approximation results for a single agent automatically improve the approximation bounds for related multi-agent mechanism design problems based on uniform pricing, e.g., from Alaei et al. [10]. As one example, for selling an object that can be configured on sale in one of m configurations to n agents with independently (but non-identically) distributed values for each configuration (also satisfying a regularity property), the second-price auction with an anonymous reserve that configures the object as the winner most prefers is a $2e^2 \approx 14.8$ approximation to the optimal auction (which is sometimes randomized; improved from 32).

This work is part of a central area of study at the intersection of computer science and economics that aims to quantify the performance of simple, practical mechanisms versus optimal mechanisms (see Hartline [11], for a survey). Immediately related results in this area fit in to three broad categories: (i) anonymous and discriminatory reserve pricing [1], (ii) while-supplies-last posted pricing [12], and (iii) increased competition with symmetric [3] and asymmetric agents [1]. The four approximation of Hartline and Roughgarden [1] for anonymous reserve pricing is a corollary of their result for (iii) on (i). In comparison this paper considers (ii), foremost, and obtain a lower bound for (i) as a corollary.

The field of algorithmic mechanism design contains many questions of constant approximation where tight bounds are not known. A key challenge of these problems is that the worst-case bounds are not given by small instances, e.g., $n = 2$ agents, but are instead approached in the limit with n. The field lacks general methods for analysis of this kind of problem. Our approach is similar to the recent successful approach of Chen et al. [13] which identified the prior-free approximation ratio for digital good auctions as 2.42 (also in the limit with n, matching the lower bound of Goldberg et al. [14]). The approach writes the approximation ratio as the value of a mathematical program. With two success stories for this

approach in algorithmic mechanism design, we are optimistic about the development of a set of tools for analyzing worst case approximation factors and that these tools will be useful for making progress on many other similar open questions in the area.

2. PRELIMINARIES AND NOTATIONS

Revenue curves and regular instances: Each agent i has value drawn from a distribution F_i with cumulative distribution function (CDF) denoted by $F_i(\cdot)$. The *revenue curve* $R_i(q) = q \cdot F_i^{-1}(1-q)$ gives the expected revenue obtained by selling an item to agent i with probability exactly q, i.e., by posting price $F^{-1}(1-q)$. The agent is *regular* if its revenue curve $R_i(q)$ is concave in q. An n-agent instance $\mathcal{I} = \{F_i\}_{i=1}^n$ is regular if each agent's distribution is regular. The family of all regular instances for all $n \geq 1$ is denoted by REG.

Anonymous pricings: An anonymous pricing is a mechanism that posts a price p that is bought by an arbitrary agent whose value is at least the posted price (if one exists). The expected revenue of the anonymous pricing p for instance $\mathcal{I} = \{F_i\}_{i=1}^n$ is

$$\text{PRICEREV}(\mathcal{I}, p) \triangleq p \cdot \left(1 - \prod_i F_i(p)\right). \tag{1}$$

The expected revenue of the optimal anonymous pricing is

$$\text{OPTPRICEREV}(\mathcal{I}) \triangleq \max_{p \in \mathbb{R}_+} \text{PRICEREV}(\mathcal{I}, p). \tag{2}$$

Ex ante relaxation and optimal auctions: The revenue of the *ex ante relaxation*, which allocates to one agent in expectation, gives an upper bound on the revenue of the optimal auction. For any instance $\mathcal{I}$, it can be easily expressed in terms of the revenue curves of the agents.

$$
\begin{aligned}
\text{EXANTEREV}(\mathcal{I}) \triangleq \quad & \max \quad \sum_{i=1}^n R_i(q_i) \\
& \text{subject to} \quad \sum_{i=1}^n q_i \leq 1 \\
& q_i \geq 0 \qquad \forall i \in \{1, \ldots, n\}.
\end{aligned}
\tag{3}
$$

Note: We would prefer to compare the performance of anonymous pricing directly to the optimal auction of Myerson [2]; however, the standard formulation of the expected revenue of the optimal mechanism is difficult to analyze relative to the optimal anonymous pricing.

Worst-case approximation ratio: The main task of this paper is to analyze the worst case ratio of the revenue of the ex ante relaxation to the revenue of the optimal anonymous pricing over all regular instances, that is

$$\rho \triangleq \sup_{\mathcal{I} \in \text{REG}} \frac{\text{EXANTEREV}(\mathcal{I})}{\text{OPTPRICEREV}(\mathcal{I})}, \tag{P1}$$

where REG denotes the space of all regular instances.

Triangular revenue curve instances: We will show that distributions with triangular-shaped revenue curves give worst case instances for program (P1). A *triangular revenue curve* distribution, denoted $\text{Tri}(\bar{v}, \bar{q})$ with parameters $\bar{v} \in (0, \infty)$ and $\bar{q} \in [0, 1]$, has CDF given by

$$F(p) = \begin{cases} 1 & p \geq \bar{v} \\ \frac{p \cdot (1-\bar{q})}{p \cdot (1-\bar{q}) + \bar{v}\bar{q}} & 0 \leq p < \bar{v} \end{cases} \qquad \forall p \in \mathbb{R}_+. \tag{4}$$

The revenue curve corresponding to the above distribution has the form of a triangle with vertices at $(0, 0)$, $(\bar{q}, \bar{v}\bar{q})$, and $(1, 0)$ as illustrated in Fig. 2; the revenue curve's concavity implies that the distribution is regular. Note that the CDF is discontinuous at $\bar{v}$ which corresponds to a pointmass of $\bar{q}$ at value $\bar{v}$. A
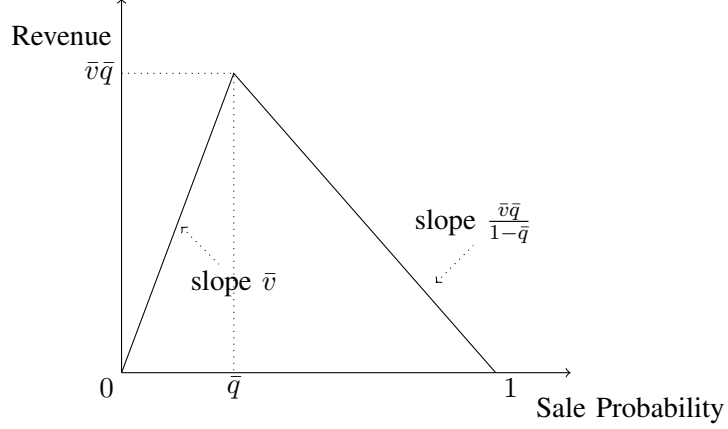
Figure 2.   Revenue curve of distribution $\mathrm{Tri}(\bar{v}, \bar{q})$.

triangular revenue curve instance is given by $\mathcal{I} = \{\mathrm{Tri}(\bar{v}_i, \bar{q}_i)\}_{i=1}^n$ with $\sum_{i=1}^n \bar{q}_i \le 1$; with respect to it the revenue of anonymous pricing p and the ex ante relaxation are given by

$$\text{PRICEREV}(\mathcal{I}, p) = p \cdot \left(1 - \prod_{i:\bar{v}_i \ge p} \left(1 + \frac{\bar{v}_i \bar{q}_i}{p \cdot (1 - \bar{q}_i)}\right)^{-1}\right), \tag{5}$$

$$\text{EXANTEREV}(\mathcal{I}) = \sum_{i=1}^n \bar{v}_i \bar{q}_i. \tag{6}$$

3. Upper-Bound Analysis

Program (P1) defines a tight upper bound on the ratio, denoted by ρ, of the revenue of the ex ante relaxation to the revenue of the optimal anonymous pricing. This program can be thought of as a continuous optimization problem over regular distributions with the objective of maximizing the aforementioned ratio. This section shows the upper bound of Theorem 1.1 while Section 4 shows tightness.

Overview of the analysis: By normalizing the optimal anonymous pricing revenue to be one, (P1) is equivalent to the following program:

$$\rho = \sup_{\mathcal{I} \in \text{REG}} \quad \text{EXANTEREV}(\mathcal{I}) \tag{P2}$$

$$\text{subject to} \quad \text{PRICEREV}(\mathcal{I}, p) \le 1 \qquad \forall p \ge 1. \tag{P2.1}$$

Note that $\text{PRICEREV}(\mathcal{I}, p) < 1$ for $p \in [0, 1)$, so it is safe to assume prices are in range $[1, +\infty)$. We show that for any fixed n the supremum of this program is approached even when restricting to triangular revenue curve instances, i.e., ones of the form $\{\mathrm{Tri}(\bar{v}_i, \bar{q}_i)\}_{i=1}^n$ with $\sum_i \bar{q}_i \le 1$ as defined in Section 2. Consequently, the problem is reduced to a discrete optimization problem over variables $\bar{\mathbf{v}} \triangleq (\bar{v}_1, \dots, \bar{v}_n)$ and $\bar{\mathbf{q}} \triangleq (\bar{q}_1, \dots, \bar{q}_n)$. An *assignment* for this optimization problem refers to a pair $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$. This optimization problem is still of infinite dimension because n is itself a variable. It also turns out to be highly non-convex. Re-index $\bar{\mathbf{v}}$ such that $\bar{v}_1 \ge \dots \ge \bar{v}_n$. We will show that, for any fixed n, inequality (P2.1) can be assumed without loss of generality to be tight for all $p \in \{\bar{v}_1, \dots, \bar{v}_n\}$; otherwise an instance for which at least one of these constraints is not tight could be modified to make all these constraints tight while improving the objective. Thus,

$$\text{PRICEREV}(\{\mathrm{Tri}(\bar{v}_i, \bar{q}_i)\}_{i=1}^n, \bar{v}_k) = 1 \qquad\qquad \forall k \in \{1, \dots, n\}. \tag{7}$$

Observe that for each $k \in \{1, \dots, n\}$ the left hand side of equation (7) only depends on the first k agents, because the valuations of the rest of the agents are always below $\bar{v}_k$. Consequently once $\bar{v}_1, \dots, \bar{v}_n$ are fixed, we can compute $\bar{q}_1, \dots, \bar{q}_n$ by solving equation (7) for $k \in \{1, \dots, n\}$ and using

forward substitution. Unfortunately, the resulting formulation of $\bar{q}_k$ in terms of $\bar{v}_1, \ldots, \bar{v}_k$ is difficult to analyze directly for $k \geq 2$. Instead, we relax inequality (P2.1) in such a way that it leads to a tractable formulation of $\bar{\mathbf{q}}$ in terms of $\bar{\mathbf{v}}$. We also show that the relaxed inequality is tight which implies the value of the relaxed program is equal to that of the original program. Finally, we show that the supremum of the relaxed program is attained when $n \to \infty$, and roughly speaking the instance converges to a continuum of infinitesimal agents with triangular revenue curve distributions. For this continuum of agents, ρ is given simply by the optimization of p in the objective $1 + \mathcal{V}(p)$ subject to the constraint $\mathcal{Q}(p) \leq 1$ for the two functions $\mathcal{V}(\cdot)$ and $\mathcal{Q}(\cdot)$ given in the statement of Theorem 1.1.

Reduction to triangular revenue curve instances: We begin by showing that without loss of generality we can restrict program (P2) to triangle revenue curve instances.

Lemma 3.1. *The supremum of program* (P2) *is approached by triangle revenue curve instances, i.e., of the form* $\hat{\mathcal{I}} = \{\mathrm{Tri}(\bar{v}_i, \bar{q}_i)\}_{i=1}^n$ *with* $\sum_{i=1}^n \bar{q}_i \leq 1$.

Proof: We will show that for any regular instance $\mathcal{I} = \{F_i\}_{i=1}^n$, there exists a corresponding instance $\hat{\mathcal{I}} = \{\mathrm{Tri}(\bar{v}_i, \bar{q}_i)\}_{i=1}^n$ with $\sum_{i=1}^n \bar{q}_i \leq 1$ yielding the same optimal ex ante revenue and (weakly) smaller expected revenue from the optimal anonymous price.

Let $\bar{\mathbf{q}}$ be an optimal assignment for the ex ante relaxation program (3) that computes EXANTEREV($\mathcal{I}$). Set $\bar{v}_i \leftarrow R_i(\bar{q}_i)/\bar{q}_i$ for each $i \in \{1, \ldots, n\}$, where R_i is the revenue curve of F_i. We show changing agent i's valuation distribution to $\mathrm{Tri}(\bar{v}_i, \bar{q}_i)$ can only decrease the revenue of any anonymous pricing (PRICEREV) while preserving the revenue of the ex ante relaxation (EXANTEREV), which implies the statement of the lemma.

Let $\hat{R}_i$ be the revenue curve of $\mathrm{Tri}(\bar{v}_i, \bar{q}_i)$. Observe that the change of distributions does not affect EXANTEREV because $\hat{R}_i(\bar{q}_i) = R_i(\bar{q}_i)$ for all $i \in \{1, \ldots, n\}$, and $\hat{R}_i$ is a lower bound on R_i elsewhere as R_i is concave (see Fig. 3). Therefore the replacement preserves the optimal value of convex program of Eq. (3) which implies EXANTEREV($\hat{\mathcal{I}}$) = EXANTEREV($\mathcal{I}$).
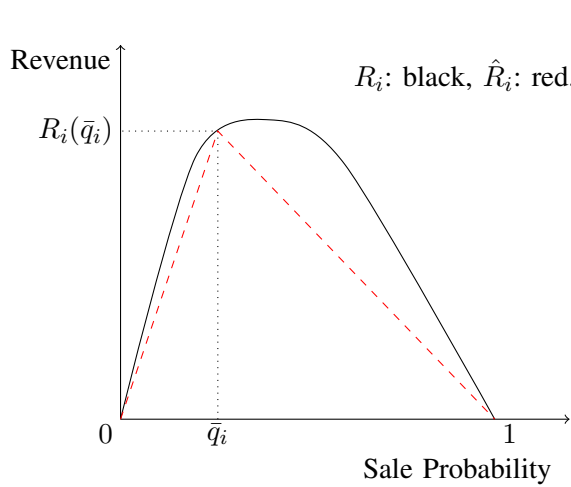

Figure 3. Replacing regular distribution F_i with triangular revenue curve distribution $\mathrm{Tri}(\bar{v}_i, \bar{q}_i)$.
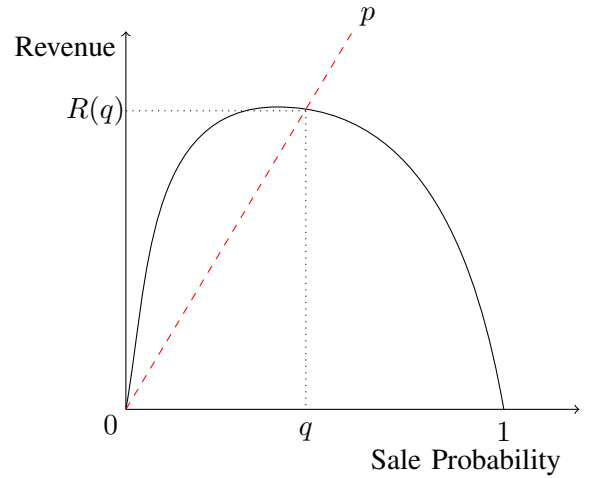
Figure 4. Intersection of revenue curve and price line p.

Next, we show the replacement may only decrease the value of PRICEREV(p) at any $p > 0$. Fix a price p, and consider the price line corresponding to p, that is, the line with slope p passing through the origin (see Fig. 4). Observe that the probability of agent i's valuation being above p is equal to the q at which $R_i(q)$ intersects price line p. Given that $\hat{R}_i$ is a lower bound on R_i everywhere, the replacement

may only decrease the probability of agent i's valuation being above p. Consequently, given that agents' valuations are distributed independently, the replacement may only decrease the revenue from sale at any anonymous price p, which implies $\textsc{PriceRev}(\hat{\mathcal{I}}) \leq \textsc{PriceRev}(\mathcal{I})$. ∎

Combining Lemma 3.1 with the formulation of $\textsc{PriceRev}$ and $\textsc{ExAnteRev}$ from Eqs. (5) and (6) yields the following non-convex program for computing ρ:

$$\rho = \sup_{n \in \mathbb{N}, \bar{\mathbf{v}}, \bar{\mathbf{q}}} \sum_{i=1}^{n} \bar{v}_i \bar{q}_i \tag{P3}$$

$$\text{subject to} \quad p \cdot \left(1 - \prod_{i : \bar{v}_i \geq p} \frac{1}{1 + \frac{\bar{v}_i \bar{q}_i}{p \cdot (1 - \bar{q}_i)}} \right) \leq 1, \qquad \forall p \geq 1 \tag{P3.1}$$

$$\sum_{i=1}^{n} \bar{q}_i \leq 1$$

$$\bar{v}_i \geq 0, \bar{q}_i \geq 0 \qquad \forall i \in \{1, \ldots, n\}.$$

Relaxations and canonical assignments: In this section we find a relaxation of program (P3) where the corresponding *pricing revenue constraint* (P3.1) is tight for all $p \in \{\bar{v}_1, \ldots, \bar{v}_n\}$ and can thus be written as a program on variables $\bar{\mathbf{v}}$ alone (i.e., by solving for the appropriate $\bar{\mathbf{q}}$ in terms of $\bar{\mathbf{v}}$). To simplify the solution of $\bar{\mathbf{q}}$ in terms of $\bar{\mathbf{v}}$, we will first make a series of relaxations to the pricing revenue constraint (P3.1). We will point which of these relaxations are obviously tight, and the others we will prove to be tight in the limit with the number of agents n in Section 4, where we derive the matching lower bound.

Lemma 3.2 formalizes these relaxations as sketched below, the formal proof is given in the appendix. First, observe that the pricing revenue constraint (P3.1) can be rearranged as

$$\prod_{i : \bar{v}_i \geq p} \left(1 + \frac{\bar{v}_i \bar{q}_i}{p \cdot (1 - \bar{q}_i)} \right) \leq \left(\frac{p}{p - 1} \right) \qquad \forall p \geq 1.$$

The first relaxation drops the constraint on $p \notin \{\bar{v}_1, \ldots, \bar{v}_n\}$; this is without loss as the optimal anonymous price is always in $\{\bar{v}_1, \ldots, \bar{v}_n\}$. We re-index such that $\bar{v}_1 \geq \cdots \geq \bar{v}_n$ and rephrase the relaxed constraint as

$$\prod_{i=1}^{k} \left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k \cdot (1 - \bar{q}_i)} \right) \leq \left(\frac{\bar{v}_k}{\bar{v}_k - 1} \right) \qquad \forall k \in \{1, \ldots, n\}.$$

As the second relaxation we drop the term $(1 - \bar{q}_i)$ from the denominator of the left hand side and take the logarithm of both sides to get

$$\sum_{i=1}^{k} \ln \left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k} \right) \leq \ln \left(\frac{\bar{v}_k}{\bar{v}_k - 1} \right) \qquad \forall k \in \{1, \ldots, n\}.$$

As the third relaxation we upper-bound $\ln \left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k} \right)$ by $\frac{1}{\bar{v}_k} \ln(1 + \bar{v}_i \bar{q}_i)$ for $i \geq 2$. Rearranging gives

$$\sum_{i=2}^{k} \ln \left(1 + \bar{v}_i \bar{q}_i \right) \leq \bar{v}_k \ln \left(\frac{\bar{v}_k^2}{(\bar{v}_k - 1)(\bar{v}_k + \bar{v}_1 \bar{q}_1)} \right) \qquad \forall k \in \{2, \ldots, n\}.$$

The previous relaxation uses the fact that $\frac{1}{a} \ln(1 + b) \leq \ln(1 + \frac{b}{a})$ for all $a \geq 1, b \geq 0$. In the proof, we will also show that $\bar{v}_1 \bar{q}_1$ can be replaced with 1 both in the above constraint and in the objective

function without loss of generality. Putting everything together, we will obtain the following program as a relaxation of program (P3).

$$\rho' = \sup_{n \in \mathbb{N}, \bar{\mathbf{v}}, \bar{\mathbf{q}}} \quad 1 + \sum_{i=2}^{n} \bar{v}_i \bar{q}_i \tag{P4}$$

$$\text{subject to} \quad \sum_{i=2}^{k} \ln\left(1 + \bar{v}_i \bar{q}_i\right) \leq \mathcal{V}(\bar{v}_k) \qquad \forall k \in \{2, \ldots, n\} \tag{P4.1}$$

$$\sum_{i=2}^{n} \bar{q}_i \leq 1$$

$$\bar{v}_{i+1} \leq \bar{v}_i, \qquad \forall i \in \{2, \ldots, n-1\}$$

$$\bar{v}_i \geq 0, \bar{q}_i \geq 0 \qquad \forall i \in \{1, \ldots, n\}.$$

where $\mathcal{V}(\cdot) = p \cdot \ln\left(\frac{p^2}{p^2-1}\right)$ is defined in Theorem 1.1.

Lemma 3.2. *The value of program* (P4)*, denoted by ρ', is an upper bound on the value of program* (P3) *which is ρ.*

Next we show that we can assume without loss of generality the pricing revenue constraint (P4.1) is tight for all $k \in \{2, \ldots, n\}$ in program (P4). That will allow us to specify one set of variables (e.g., $\bar{\mathbf{q}}$) in terms of the other set of variables (e.g., $\bar{\mathbf{v}}$), which consequently allows us to eliminate the former variables and drop the pricing revenue constraint (P4.1). To this end, we first define a *canonical* feasible solution for (P4), restriction to which is without loss as given by Lemma 3.3.

Definition 3.1. A feasible assignment $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ for (P4) is *canonical* if the pricing constraint (P4.1) is tight for all $k \in \{2, \ldots, n\}$.

Lemma 3.3. *For any feasible assignment $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ for* (P4)*, there exists an equivalent canonical feasible assignment $(\bar{\mathbf{v}}', \bar{\mathbf{q}}')$ obtaining the same objective value, that is $\sum_i \bar{v}_i \bar{q}_i = \sum_i \bar{v}'_i \bar{q}'_i$.*

Proof: Without loss of generality assume $\bar{q}_k > 0$ for all $k \in \{2, \ldots, n\}$.[2] The right hand side of the pricing constraint (P4.1) is $\mathcal{V}(\bar{v}_k)$ which is decreasing in $\bar{v}_k$ (see Lemma 3.4) and approaches 0 as $\bar{v}_k \to \infty$, so for every $k \in \{2, \ldots, n\}$ there exists $\bar{v}'_k \geq \bar{v}_k$ such that

$$\sum_{i=2}^{k} \ln\left(1 + \bar{v}_i \bar{q}_i\right) = \mathcal{V}(\bar{v}'_k) \qquad \forall k \in \{2, \ldots, n\}.$$

Observe that by the above construction we always have $\bar{v}'_2 \geq \ldots \geq \bar{v}'_n$. We then decrease $\bar{q}_k$ to $\bar{q}'_k = \bar{q}_k \frac{\bar{v}_k}{\bar{v}'_k}$ for each $k \in \{2, \ldots, n\}$ to obtain the desired assignment $(\bar{\mathbf{v}}', \bar{\mathbf{q}}')$. ∎

By Lemma 3.3, we can restrict our attention to canonical assignments of (P4) without loss of generality. In particular, we can fully identify such a canonical assignment by specifying only $\bar{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_n)$ since the corresponding $\bar{\mathbf{q}}$ is given by

$$\bar{q}_k = \frac{e^{\mathcal{V}(\bar{v}_k) - \mathcal{V}(\bar{v}_{k-1})} - 1}{\bar{v}_k} \qquad \forall k \in \{2, \ldots, n\}. \tag{8}$$

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[2]If $\bar{q}_k = 0$, we can drop agent k without affecting feasibility or objective value.

Therefore we can obtain from program (P4) the following program.

$$\rho' = \sup_{n \in \mathbb{N}, \bar{\mathbf{v}}} \quad 1 + \sum_{i=2}^{n} \bar{v}_i \bar{q}_i \tag{P5}$$

$$\text{subject to} \quad \bar{q}_k = \frac{e^{\mathcal{V}(\bar{v}_k) - \mathcal{V}(\bar{v}_{k-1})} - 1}{\bar{v}_k} \qquad \forall k \in \{2, \dots, n\} \tag{P5.1}$$

$$\sum_{i=2}^{n} \bar{q}_i \leq 1 \tag{P5.2}$$

$$\bar{v}_{i+1} \leq \bar{v}_i, \qquad \qquad \forall i \in \{2, \dots, n-1\}$$

$$\bar{v}_i \geq 0, \ \bar{q}_i \geq 0 \qquad \qquad \forall i \in \{1, \dots, n\}.$$

Continuum of agents: Given that n itself is a variable, a solution to program (P5) can be practically specified by a finite subset $\bar{\mathbf{v}} \subset \mathbb{R}_+$ where $\bar{v}_i$ is the ith largest value in that subset. We now show that the optimal solution to program (P5) corresponds to $\bar{\mathbf{v}} = [p, \infty)$ (for some $p > 1$) which can be viewed as an instance with infinitely many infinitesimal agents (i.e., with $\bar{q}_i$ going to zero).

For any given $p' > p > 1$, we define a continuum of agents $[p, p')$ by defining for each $m \in \mathbb{N}$ a discrete family of agents of size m spanning $[p, p')$ and by taking the limit of this family as $m \to \infty$. Formally, for each $m \in \mathbb{N}$, we consider the family of agents with distributions $\{\text{Tri}(u_j, (e^{\mathcal{V}(u_j) - \mathcal{V}(u_{j-1})} - 1)/u_j)\}_{j=1}^{m}$ where $u_j = p' + \frac{j}{m}(p - p')$. Observe that the agents in these families satisfy equation (8). Furthermore, observe that

$$\lim_{\delta \to 0} \left(\frac{e^{\mathcal{V}(v) - \mathcal{V}(v+\delta)} - 1}{v} \cdot \frac{1}{\delta} \right) = -\frac{\mathcal{V}'(v)}{v}.$$

Therefore in a continuum of agents $[p, p')$ each infinitesimal agent $v \in [p, p')$ has a distribution of $\text{Tri}(v, -\frac{\mathcal{V}'(v)}{v} \, dv)$, which implies that the contribution of $[p, p')$ to the objective value of (P5) is

$$\int_{p}^{p'} v \cdot (-\frac{\mathcal{V}'(v)}{v}) \, dv = \mathcal{V}(p) - \mathcal{V}(p'), \tag{9}$$

and the contribution of $[p, p')$ to the left hand side of the constraint (P5.2), i.e. $\sum_i \bar{q}_i \leq 1$ which is referred to as the *capacity constraint*, is

$$\int_{p}^{p'} -\frac{\mathcal{V}'(v)}{v} \, dv = \mathcal{Q}(p) - \mathcal{Q}(p'), \tag{10}$$

where $\mathcal{Q}(p) = \int_{p}^{\infty} -\frac{1}{v} \mathcal{V}'(v) \, dv$ as defined in Theorem 1.1.

Via the above derivation of a continuum of agents, program (P5), on the instance corresponding to the continuum $[p, \infty)$, simplifies as:

$$\rho'' = \max_{p \geq 1} \quad 1 + \mathcal{V}(p) \tag{P6}$$

$$\text{subject to} \quad \mathcal{Q}(p) \leq 1. \qquad .$$

Next we will sketch a construction that demonstrates that any feasible solution $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_n)$ to program (P5) can be replaced by a continuum of agents that corresponds to an interval $[p, \infty)$ (for some $p > 1$ to be determined) and the objective of (P5) is strictly increased. Note that $\bar{v}_1$ does not appear anywhere in (P5); for notational convenience we redefine it as $\bar{v}_1 = \infty$. Suppose for each $i \in \{2, \dots, n\}$ we replace the agent $\bar{v}_i$ with the continuum of agents $[\bar{v}_i, \bar{v}_{i-1})$. It follows from Eqs. (8) and (9) that this replacement changes the object value of (P5) by $\mathcal{V}(\bar{v}_i) - \mathcal{V}(\bar{v}_{i-1}) - \bar{v}_i \bar{q}_i = \ln(1 + \bar{v}_i \bar{q}_i) - \bar{v}_i \bar{q}_i < 0$ which is unfortunately always negative and thus the opposite of what we want to prove. On the other

hand, it follows from Eqs. (8) and (10) that this replacement also changes the left hand side of the capacity constraint (P5.2) by $\mathcal{Q}(\bar{v}_i) - \mathcal{Q}(\bar{v}_{i-1}) - \bar{q}_i$ which is also negative (as we will show later), and thus creates some slack in the capacity constraint (P5.2). Summing over the slack created in the capacity constraint (P5.2) from converting each agent to a continuum, we can add a new continuum of agents $[p, \bar{v}_n)$ where $p < \bar{v}_n$ is chosen to make the capacity constraint (P5.2) tight. As a consequence of the following claims, the net change in the objective value from this transformation is positive.

(i) The amount of slack created in the capacity constraint (P5.2) by replacing $\bar{v}_i$ with $[\bar{v}_i, \bar{v}_{i-1})$ is more than the decrease in the objective value of (P5). Using Eqs. (9) and (10), we can formally write this claim as $\bar{q}_i - (\mathcal{Q}(\bar{v}_i) - \mathcal{Q}(\bar{v}_{i-1})) > \bar{v}_i \bar{q}_i - (\mathcal{V}(\bar{v}_i) - \mathcal{V}(\bar{v}_{i-1}))$. This is proved below in Lemma 3.6.

(ii) If there is a slack of $\Delta > 0$ in the capacity constraint (P5.2), it can be used to extend the last continuum of agents to increase the objective value by more than Δ. Using Eqs. (9) and (10), we can formalize this claim as follows: if p is chosen such that $\mathcal{Q}(p) - \mathcal{Q}(\bar{v}_n) = \Delta$, then $\mathcal{V}(p) - \mathcal{V}(\bar{v}_n) > \Delta$. This is proved below in Lemma 3.4.

The suggestion from the above construction is that from any solution $\bar{v}$ to program (P5), a price p can be identified such that the continuum of agents on $[p, \infty)$ has higher objective value. In other words, the optimal values of program (P5) and program (P6) are equal. This is proved below in Lemma 3.7; though we defer the proof that the solution of program (P6) corresponds to a limit solution of program (P5) to Section 4.

Algebraic upper-bound proof: The rest of this section develops a formal but purely algebraic proof that is based on the approach sketched in the previous paragraphs. The proofs of the first two lemmas, below, can be found in the appendix.

Lemma 3.4. *The functions $\mathcal{V}(p)$, $\mathcal{Q}(p)$, and $\mathcal{V}(p) - \mathcal{Q}(p)$ are all decreasing in p, for $p > 1$.*

Lemma 3.5. *for any $p' > p > 1$ the following inequality holds: $\mathcal{V}(p) - \mathcal{V}(p') < \ln(\frac{p}{p-1}) - \ln(\frac{p'}{p'-1})$.*

Lemma 3.6. *For any $p' > p > 1$ and $q = \frac{e^{\mathcal{V}(p) - \mathcal{V}(p')} - 1}{p}$ the following inequalities hold:*

$$q - (\mathcal{Q}(p) - \mathcal{Q}(p')) \geq pq - (\mathcal{V}(p) - \mathcal{V}(p')) \geq 0 \tag{11}$$

Proof: Define

$$W(p, p') \triangleq \mathcal{V}(p) - \mathcal{Q}(p) - \mathcal{V}(p') + \mathcal{Q}(p') + q - pq$$
$$= \mathcal{V}(p) - \mathcal{Q}(p) - \mathcal{V}(p') + \mathcal{Q}(p') - (p-1)(e^{\mathcal{V}(p) - \mathcal{V}(p')} - 1)/p.$$

Observe that proving the first inequality in the statement of the lemma is equivalent to proving $W(p, p') > 0$. We instead prove that $W(p, p')$ is increasing in p' which together with the trivial fact that $W(p, p) = 0$ implies $W(p, p') > 0$.

$$\tfrac{\partial}{\partial p'} W(p, p') = -\mathcal{V}'(p') + \mathcal{V}'(p')/p' + (p-1)\mathcal{V}'(p') e^{\mathcal{V}(p) - \mathcal{V}(p')}/p$$
$$= -\mathcal{V}'(p') \left[\frac{p'-1}{p'} - \frac{p-1}{p} e^{\mathcal{V}(p) - \mathcal{V}(p')} \right]$$
$$> -\mathcal{V}'(p') \left[\frac{p'-1}{p'} - \frac{p-1}{p} e^{\ln(\frac{p}{p-1}) - \ln(\frac{p'}{p'-1})} \right] = 0.$$

The final inequality follows from Lemmas 3.4 and 3.5: by Lemma 3.4, $-\frac{\partial}{\partial p'}\mathcal{V}(p') > 0$; and by Lemma 3.5, $e^{\mathcal{V}(p) - \mathcal{V}(p')}$ is less than $e^{\ln(\frac{p}{p-1}) - \ln(\frac{p'}{p'-1})}$, so replacing the former with the latter only decreases the value of the expression inside the brackets because its coefficient is $-\frac{p-1}{p}$ which is negative.

The second inequality in the statement of the lemma follows trivially from the fact that $\mathcal{V}(p) - \mathcal{V}(p') = \ln(1 + pq)$ thus $pq - (\mathcal{V}(p) - \mathcal{V}(p')) = pq - \ln(1 + pq) > 0$. ∎

Lemma 3.7. *The value of program* (P6), *denoted by ρ'', is an upper bound on the value of program* (P5) *which is ρ'.*

Proof: Let $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ be any arbitrary feasible assignment for program (P5). We show there exists a feasible assignment for program (P6) with objective value upper bounding the objective value of $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ in program (P5). Define $p^* \triangleq \mathcal{Q}^{-1}(1)$, a candidate solution to program (P6) that meets the feasibility constraint with equality. Note that such a p^* exists because $\mathcal{Q}(\infty) = 0$, $\mathcal{Q}(1) = \infty$, and $\mathcal{Q}(\cdot)$ is continuous. Observe that the objective value of (P5) for $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ satisfies:

$$1 + \sum_{k=2}^{n} \bar{v}_k \bar{q}_k \leq 1 + \sum_{k=2}^{n} \big(\mathcal{V}(\bar{v}_k) - \mathcal{V}(\bar{v}_{k-1}) - (\mathcal{Q}(\bar{v}_k) - \mathcal{Q}(\bar{v}_{k-1})) + \bar{q}_k \big) \quad \text{by Lemma 3.6 and } \bar{v}_1 = \infty$$

$$= 1 + \mathcal{V}(\bar{v}_n) - \mathcal{Q}(\bar{v}_n) + \sum_{k=2}^{n} \bar{q}_k \qquad\qquad\qquad \text{as } \mathcal{V}(\infty) = \mathcal{Q}(\infty) = 0$$

$$\leq 1 + \mathcal{V}(\bar{v}_n) - \mathcal{Q}(\bar{v}_n) + 1 \qquad\qquad\qquad\quad \text{as } \sum_{k=2}^{n} \bar{q}_k \leq 1$$

$$< 1 + \mathcal{V}(p^*) - \mathcal{Q}(p^*) + 1 \qquad\qquad\qquad\qquad \text{as proved below} \qquad (*)$$

$$= 1 + \mathcal{V}(p^*) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{as } \mathcal{Q}(p^*) = 1.$$

To prove inequality $(*)$ we show that $p^* < \bar{v}_n$ which together with Lemma 3.4 implies $\mathcal{V}(p^*) - \mathcal{Q}(p^*) > \mathcal{V}(\bar{v}_n) - \mathcal{Q}(\bar{v}_n)$. To prove $p^* < \bar{v}_n$ observe that Lemma 3.6 implies $\sum_{i=2}^{n} \bar{q}_i > \sum_{i=2}^{n} \mathcal{Q}(\bar{v}_i) - \mathcal{Q}(\bar{v}_{i-1}) = \mathcal{Q}(\bar{v}_n)$. On the other hand $\sum_{i=2}^{n} \bar{q}_i \leq 1$. Therefore $\mathcal{Q}(\bar{v}_n) < 1$ which implies $\bar{v}_n > p^*$ because $\mathcal{Q}(p^*) = 1$ and, by Lemma 3.4, $\mathcal{Q}(\cdot)$ is decreasing. ∎

We conclude the section with the proof of the upper-bound of Theorem 1.1.

Proof of upper-bound in Theorem 1.1: It follows from program (P1), Lemmas 3.1 to 3.3, and the rest of the discussion in this section that ρ' which is computed by (P5) is an upper bound on the ratio of the ex ante relaxation to the expected revenue of the optimal anonymous pricing. Following Lemma 3.7, ρ' is upper bounded by the objective value of program (P6), i.e. ρ''. As $\mathcal{Q}(\cdot)$ and $\mathcal{V}(\cdot)$ are decreasing (Lemma 3.4), the optimal solution to program (P6) is given by $\rho'' = 1 + \mathcal{V}(\mathcal{Q}^{-1}(1))$ which numerically evaluates to $e \approx 2.718$. ∎

4. Lower-Bound Analysis

In this section we show the tightness of our approximation, i.e. the lower-bound in Theorem 1.1. As a result of Lemma 3.1, it suffices to prove the following lemma.

Lemma 4.1. *For any $\epsilon > 0$ there exists a feasible assignment $(n, \bar{\mathbf{v}}, \bar{\mathbf{q}})$ of the program* (P3) *such that $\sum_{i=1}^{n} \bar{v}_i \bar{q}_i \geq 1 + \mathcal{V}(\mathcal{Q}^{-1}(1)) - \epsilon$.*

Proof: Pick $\delta > 0$ s.t. $(1-\delta)^2 \left(1 + \mathcal{V}\left(\mathcal{Q}^{-1}\left(\frac{1}{(1+\delta)^2} \right) + \delta \right) \right) \geq 1 + \mathcal{V}(\mathcal{Q}^{-1}(1)) - \epsilon$. This is always possible as $\mathcal{V}$ and $\mathcal{Q}$ are decreasing. Lets define $\lambda = \mathcal{Q}^{-1}(1)$. The proof is done in two steps:

Step 1: We find $\{v_i, q_i\}_{i=2}^{n}$ such that

$$\sum_{i=2}^{n} q_i \leq 1 \quad , \quad k = 2, \ldots, n : \quad \sum_{i=2}^{k} \ln(1 + v_i q_i) = \mathcal{V}(v_k)$$

$$\sum_{i=2}^{n} v_i q_i \geq \mathcal{V}\left(\mathcal{Q}^{-1}\left(\frac{1}{(1+\delta)^2} \right) + \delta \right)$$

In our construction for $\{v_i, q_i\}_{i=2}^n$, we use two parameters $\Delta > 0$ and $V_T \geq \lambda$ which we fix later in the proof. Let $v_1 \triangleq \infty$, and for $i \geq 2$ set $v_i = V_T - (i-2)\Delta$ and $q_i = \frac{e^{\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})} - 1}{v_i}$. Now, let $n = \max\{n_0 \in \mathbb{N} : \sum_{i=2}^{n_0} q_i \leq 1\}$. Obviously, $\sum_{i=2}^n q_i \leq 1$. Moreover, for any $2 \leq k \leq n$ we have $\sum_{i=2}^k \ln(1 + v_i q_i) = \sum_{i=2}^k (\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})) = \mathcal{V}(v_k) - \mathcal{V}(v_1) = \mathcal{V}(v_k)$. Now, pick $\delta' > 0$ small enough such that for $x \in [0, \delta']$ we have $\frac{e^x - 1}{x} \leq 1 + \delta$. Moreover, let Δ to be small enough and V_T to be large enough such that $\max\{\mathcal{V}(\lambda) - \mathcal{V}(\lambda + \Delta), \mathcal{V}(V_T), \Delta\} \leq \min\{\delta, \delta'\}$. First observe that due to Lemma 3.6 $q_i \geq \mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1})$ which implies $2 \geq \sum_{i=1}^n q_i \geq \mathcal{Q}(v_n) - \mathcal{Q}(v_1) = \mathcal{Q}(v_n)$. So, all values v_i are at least equal to λ. As $\mathcal{V}(.)$ is convex over $[1, \infty)$, we have $\mathcal{V}(v_i) - \mathcal{V}(v_{i-1}) \leq \mathcal{V}(\lambda) - \mathcal{V}(\lambda + \Delta) \leq \delta'$. As a result we have

$$q_i = \frac{e^{\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})} - 1}{v_i} \leq (1 + \delta)\frac{\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})}{v_i} = (1 + \delta)\int_{v_i}^{v_{i-1}} \frac{\mathcal{V}'(v)}{v_i} \, dv.$$

$$= (1 + \delta)\int_{v_i}^{v_{i-1}} -\frac{v}{v_i}\mathcal{Q}'(v) \, dv = (1 + \delta)\left(\mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1}) + \int_{v_i}^{v_{i-1}} -\frac{v - v_i}{v_i}\mathcal{Q}'(v) \, dv\right)$$

$$\leq (1 + \delta)\left(\mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1}) + \Delta \int_{v_i}^{v_{i-1}} -\mathcal{Q}'(v) \, dv\right) \leq (1 + \delta)^2(\mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1})) \qquad (12)$$

Based on the definition of n (number of distributions in our instance), we have $1 < \sum_{i=2}^{n+1} q_i$. By (12), we have $\sum_{i=2}^{n+1} q_i \leq (1 + \delta)^2 \sum_{i=2}^{n+1}((\mathcal{Q}(v_i) - \mathcal{Q}(v_{i-1})) = (1 + \delta)^2 \mathcal{Q}(v_{n+1})$. Lets define $\lambda' \triangleq \mathcal{Q}^{-1}\left(\frac{1}{(1+\delta)^2}\right)$. We conclude that $\lambda' \geq v_{n+1}$. Hence, $v_n \leq \lambda' + \Delta \leq \lambda' + \delta$. Moreover, using Lemma 3.1 we have

$$\sum_{i=2}^n v_i q_i \geq \sum_{i=2}^n (\mathcal{V}(v_i) - \mathcal{V}(v_{i-1})) = \mathcal{V}(v_n) \geq \mathcal{V}(\lambda' + \delta) = \mathcal{V}\left(\mathcal{Q}^{-1}\left(\frac{1}{(1+\delta)^2}\right) + \delta\right) \qquad (13)$$

where the last inequality is true because $v_n \leq \lambda' + \delta$ and $\mathcal{V}$ is decreasing over $[1, \infty)$.

Step 2: Given $\{v_i, q_i\}_{i=2}^n$, we find an instance $\{\bar{v}_i, \bar{q}_i\}_{i=1}^n$ such that is feasible for program (P3) and $\sum_{i=1}^n \bar{v}_i \bar{q}_i \geq 1 + \mathcal{V}(\mathcal{Q}^{-1}(1)) - \epsilon$. To do so, set $q_1 = \delta, v_1 = \frac{1}{\delta} - 1$. Now, for each $i, k \in [2 : n]$ find $\gamma_{i,k}$ such that

$$1 + \frac{v_i q_i(1 - \gamma_{i,k})}{v_k} = (1 + v_i q_i)^{\frac{1}{v_k}} \qquad (14)$$

and then let $\bar{q}_i = (1 - \delta)(1 - \max_{k \in [2:n]} \gamma_{i,k})q_i$ and $\bar{v}_i = v_i$, for $i \in [2 : n]$. Now we claim $\{\bar{v}_i, \bar{q}_i\}_{i=1}^n$ is a feasible assignment for the program (P3) . We have

$$\sum_{i=1}^n \bar{q}_i = \delta + (1 - \delta)\sum_{i=2}^n (1 - \max_{k \in [2:n]} \gamma_{i,k})q_i \leq \delta + (1 - \delta)\sum_{i=2}^n q_i \leq 1. \qquad (15)$$

as $\sum_{i=2}^n q_i \leq 1$. Moreover, for $k \in [2 : n]$ we have

$$\sum_{i=1}^k \ln\left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k(1 - \bar{q}_i)}\right) \leq \ln\left(1 + \frac{\bar{v}_1 \bar{q}_1}{\bar{v}_k(1 - \bar{q}_1)}\right) + \sum_{i=2}^k \ln\left(1 + \frac{v_i q_i(1 - \gamma_{i,k})}{v_k}\right)$$

$$= \ln\left(\frac{v_k + 1}{v_k}\right) + \frac{1}{v_k}\sum_{i=2}^k \ln(1 + v_i q_i)$$

$$\leq \ln\left(\frac{v_k + 1}{v_k}\right) + \ln\left(\frac{v_k^2}{v_k^2 - 1}\right)$$

$$= \ln\left(\frac{\bar{v}_k}{\bar{v}_k - 1}\right)$$

By taking exponents from both sides and rearranging the terms it is not hard to see $(n, \bar{\mathbf{v}}, \bar{\mathbf{q}})$ is a feasible assignment of program (P3). Additionally, for a fixed V_T all of the v_i's are bounded, i.e. $1 \le v_i \le V_T$. So, as Δ goes to zero we have $v_i q_i \to 0$ as $q_i \to 0$, and the left hand side of (14) converges to its right hand side. As a result, for small enough Δ, we can guarantee $\gamma_{i,k} \le \delta$ for all i, k, and hence $\bar{q}_i \ge (1-\delta)^2 q_i$. So

$$\sum_{i=1}^{n} \bar{v}_i \bar{q}_i \ge (1-\delta) + (1-\delta)^2 \left(\sum_{i=2}^{n} v_i q_i \right)$$

$$\ge (1-\delta)^2 \left(1 + \sum_{i=2}^{n} v_i q_i \right)$$

$$\ge (1-\delta^2) \left(1 + \mathcal{V} \left(\mathcal{Q}^{-1} \left(\tfrac{1}{(1+\delta)^2} \right) + \delta \right) \right)$$

which implies $\sum_{i=1}^{n} \bar{v}_i \bar{q}_i \ge 1 + \mathcal{V}(\mathcal{Q}^{-1}(1)) - \epsilon$, as desired. ∎

5. IRREGULAR INAPPROXIMABILITY RESULTS

In this section we show that anonymous pricing and anonymous reserves are a tight n approximation to the optimal auction and ex ante relaxation. Specifically, we show a lower bound on the approximation factor of anonymous reserves to the optimal auction and an upper bound on the approximation factor of anonymous pricing to the ex ante relaxation. The ordering of these mechanisms by revenue then implies that all bounds are optimal and tight.

Proposition 5.1. *For n-agent, independent, non-identical, and irregular distributions the second-price auction with anonymous reserves is at best an n approximation to the optimal single-item auction.*

Proof: Consider the following value distribution

$$v_i = \begin{cases} h^i & \text{with probabiliy } h^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

On this distribution the ex ante relaxation has revenue $\sum_{i=0}^{n} h^i h^{-i} = n$ (and the optimal auction is no better). On the other hand, anonymous reserve and anonymous pricing of h^i for any $i \in \{1, \ldots, n\}$ gives revenue at least one. We will show that in the limit as h approaches infinity; these bounds are tight.

We first argue that in the limit of h the optimal auction revenue is n (the same as the ex ante relaxation). Consider the expected revenue of the following sequential posted pricing mechanism, which gives a lower-bound on the optimal revenue.[3] In decreasing order of price and until the first agent accepts her offered price, offer each agent i price h^i. This mechanism's revenue can be calculated as:

$$h^n \cdot \tfrac{1}{h^n} + \sum_{i=2}^{n} h^{n-i+1} \cdot \tfrac{1}{h^{n-i+1}} \prod_{j=1}^{i-1} (1 - \tfrac{1}{h^{n-j+1}}) = 1 + \sum_{i=2}^{n} \prod_{j=1}^{i-1} (1 - \tfrac{1}{h^{n-j+1}})$$

which converges to n as h goes to infinity.

We now prove that the expected revenue of the second-price auction with any anonymous reserve in $\{h^i\}_{i=1}^{n}$ is at most one in the limit. (Any other reserve is only worse.) In the second-price auction with reserve, the winner pays the maximum of the highest agent value below her value and the reserve. An

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[3]In fact, this sequential posted pricing mechanism is the optimal auction, but its optimality is unnecessary for the proof so we omit the details.

upper bound of this revenue is the sum of such a payment over all agents with values at least the reserve. So, for reserve h^i the contribution of $j \geq i$ to this upper bound is at most:

$$h^{-j}(j - i + h^i). \tag{16}$$

The first term, above, is the probability that agent j has high value h^j. Conditioned on her having the high value h^j, the second term bounds the agent's payment, the expected maximum of the highest lower-valued agent and the reserve h^i. It is at most $j - i + h^i$ as each agent between i and j has expected value one and the expectation of their maximum is at most the sum of their expectations. It follows from equation (16) that in the limit with h, the contribution from agent i to this bound is one and the contribution from agent $j \neq i$ is zero. Thus, the expected revenue of the second-price auction with reserve h^i is at most one in the limit. $\blacksquare$

Proposition 5.2. *For independent, non-identical, irregular n-agent single-item environments, anonymous pricing is at worst an n approximation to the ex ante relaxation.*

Proof: Define $\{(\bar{v}_i, \bar{q}_i)\}_{i=1}^n$ as in equation (6) in Section 2 where the ex ante relaxation posts price $\bar{v}_i$ to agent i which is accepted with probability $\bar{q}_i$ and has total revenue $\sum_{i=1}^n \bar{v}_i \bar{q}_i$. For any i the anonymous pricing that posts price $\bar{v}_i$ obtains at least revenue $\bar{v}_i \bar{q}_i$. Thus, picking a uniformly random price from $\{\bar{v}_i\}_{i=1}^n$ gives an n approximation to the ex ante relaxation revenue $\sum_{i=1}^n \bar{v}_i \bar{q}_i$. The optimal anonymous price is no worse. $\blacksquare$

6. SIMULATION RESULTS

In this section, we briefly discuss simulation results for the worst-case instance derived in section Section 3. From these simulations we will see how fast, as a function of the number n of agents, the worst-case ratio of the ex ante relaxation to the expected revenue of optimal anonymous pricing converges to e. Moreover, for these worst-case instances we will be able to evaluate the approximation of the optimal auction by anonymous reserves and pricing.

We find a discrete approximation to the continuum instance of the upper bound as follows. As described in Section 3, it is without loss to assume that every anonymous pricing obtains the same revenue. For any $\{\bar{q}_i\}_{i=1}^n$ that sum to one, a decreasing sequence $\{\bar{v}_i\}_{i=1}^n$ can be identified inductively as $\bar{v}_k$ can be determined from $\{(\bar{v}_i, \bar{q}_i)\}_{i=1}^{k-1}$ by binary search and revenue calculations. Any sequence of $\{\bar{q}_i\}_{i=1}^n$, thus, gives a lower bound on the ratio of the ex ante relaxation to the anonymous pricing. One such sequence is $\bar{q}_i = 1/n$ for all i. We employ a sequence in our simulations that converges faster. Specifically we set $\{\bar{q}_i\}_{i=1}^n$ as the arithmetic progression that evenly divides $[0, \frac{2}{n}]$ (it can be easily verified that this sums to one as required).

After generating the instances, we also calculate the ratio of the revenue of the optimal mechanism to the anonymous pricing revenue, and the ratio of the revenue of the optimal mechanism to the revenue of the second price with anonymous reserve mechanism for these instances. We use sampling algorithm to calculate the revenue of the second price with anonymous reserve mechanism, while the calculation of the revenue of the optimal mechanism is exact. We report the results of our simulation in Figs. 5 and 6 for various numbers n of agents.

n	2	5	10	50	100	500	1000	5000
EXANTEREV/OPTPRICEREV	2.000	2.507	2.622	2.701	2.710	2.717	2.718	2.718
OPTREV/OPTPRICEREV	2.000	2.138	2.187	2.223	2.227	2.231	2.231	2.232
OPTREV/OPTRESERVEREV	2.000	1.794	1.731	1.682	1.676	1.665	1.659	1.607

Figure 5. The ratios of the revenues of various auctions and benchmarks. Here EXANTEREV and OPTPRICEREV are the ex ante relaxation and optimal anonymous pricing revenues (as previously defined). OPTREV is the revenue of the optimal auction of Myerson [2]. OPTRESERVEREV is the revenue obtained by the second-price auction with an optimally chosen reserve price.
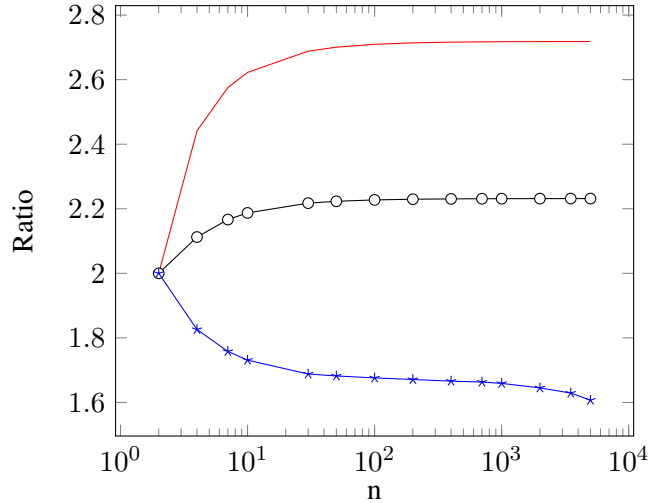
Figure 6. The ratios of the revenues of various auctions and benchmarks. The red sold line represents the ratio of ex-ante benchmark to anonymous pricing revenue. The blue line with star represents the ratio of the optimal revenue to anonymous pricing revenue. The black line with circle represents the ratio of the optimal revenue to the revenue of the second price auction with reserve.

We conclude this section by highlighting the following theorem, derived from the table in Fig. 5.

Theorem 6.1. *There exists an instance for which anonymous pricing is a 2.232 approximation to the optimal auction.*

REFERENCES

[1] J. Hartline and T. Roughgarden, "Simple versus optimal mechanisms," in *Proc. 10th ACM Conf. on Electronic Commerce*, 2009.

[2] R. Myerson, "Optimal auction design," *Mathematics of Operations Research*, vol. 6, pp. 58–73, 1981.

[3] J. Bulow and P. Klemperer, "Auctions versus negotiations," *American Economic Review*, vol. 86, pp. 180–194, 1996.

[4] S. Chawla, J. Hartline, and R. Kleinberg, "Algorithmic pricing via virtual valuations," in *Proc. 8th ACM Conf. on Electronic Commerce*, 2007.

[5] S. Alaei, "Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers," in *Proceedings of the 52nd Annual Symposium on Foundations of Computer Science*. IEEE, 2011, pp. 512–521.

[6] Q. Yan, "Mechanism design via correlation gap," in *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2011, pp. 710–719.

[7] N. Haghpanah and J. Hartline, "Reverse mechanism design," *arXiv preprint arXiv:1404.1341*, 2014.

[8] Y. Cai and C. Daskalakis, "Extreme-value theorems for optimal multidimensional pricing," in *Proceedings of the 52nd Annual Symposium on Foundations of Computer Science*. IEEE, 2011, pp. 522–531.

[9] S. Chawla, D. Malec, and B. Sivan, "The power of randomness in bayesian optimal mechanism design," in *Proc. 11th ACM Conf. on Electronic Commerce*, 2010, pp. 149–158.

[10] S. Alaei, H. Fu, N. Haghpanah, and J. Hartline, "The simple economics of approximately optimal auctions," in *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*. IEEE, 2013, pp. 628–637.

[11] J. Hartline, "Bayesian mechanism design," *Foundations and Trends® in Theoretical Computer Science*, vol. 8, no. 3, pp. 143–263, 2013.

[12] S. Chawla, J. Hartline, D. Malec, and B. Sivan, "Sequential posted pricing and multi-parameter mechanism design," in *Proc. 41st ACM Symp. on Theory of Computing*, 2010.

[13] N. Chen, N. Gravin, and P. Lu, "Optimal competitive auctions," in *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*. ACM, 2014, pp. 253–262.

[14] A. Goldberg, J. Hartline, A. Karlin, M. Saks, and A. Wright, "Competitive auctions," *Games and Economic Behavior*, vol. 55, pp. 242–269, 2006.

APPENDIX

Lemma 3.2. *The value of program* (P4)*, denoted by ρ', is an upper bound on the value of program* (P3) *which is ρ.*

Proof: Let $(\bar{\mathbf{v}}', \bar{\mathbf{q}}')$ be any feasible assignment for program (P3) for which $\sum_{i=1}^{n} \bar{v}'_i \bar{q}'_i > 1$. We construct a corresponding assignment $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ which is feasible for program (P4) and yields the same objective value, that is $1 + \sum_{i=2}^{n} \bar{v}_i \bar{q}_i = \sum_{i=1}^{n} \bar{v}'_i \bar{q}'_i$ which then implies that $\text{BOUND}_I \leq \text{BOUND}_{II}$. Without loss of generality assume $\bar{v}'_1 \geq \ldots \geq \bar{v}'_n$. Let j be the smallest index for which $\sum_{i=1}^{j} \bar{v}'_i \bar{q}'_i > 1$. Observe that $2 \leq j \leq n$ because $\bar{v}'_i \bar{q}'_i \leq 1$ for all i. Let $\delta = 1 - \sum_{i=1}^{j-1} \bar{v}'_i \bar{q}'_i$. Observe that $0 \leq \delta < \bar{v}'_j \bar{q}'_j$. We construct a new optimal assignment $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ by setting for each $i \in \{2, \ldots, n\}$:

$$\bar{v}_i = \begin{cases} \bar{v}'_{j+i-2} & 2 \leq i \leq n-j+2 \\ 1 & n-j+3 \leq i \leq n \end{cases} \qquad \bar{q}_i = \begin{cases} \bar{q}'_j - \frac{\delta}{\bar{v}'_j} & i = 2 \\ \bar{q}'_{j+i-2} & 3 \leq i \leq n-j+2 \\ 0 & n-j+3 \leq i \leq n \end{cases} \cdot$$

By the above construction it is easy to see that $1 + \sum_{i=2}^{n} \bar{v}_i \bar{q}_i = \sum_{i=1}^{n} \bar{v}'_i \bar{q}'_i$. So we only need to show that $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ is indeed a feasible assignment. Observe that $\sum_{i=2}^{n} \bar{q}_i \leq \sum_{i=1}^{n} \bar{q}'_i$, so the second constraint holds. So we only need to show that $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ satisfies the constraint (P4.1).

By rearranging (P3.1) we get

$$\prod_{i:\bar{v}'_i \geq p} \left(1 + \frac{\bar{v}'_i \bar{q}'_i}{p \cdot (1 - \bar{q}'_i)}\right) \leq \left(\frac{p}{p-1}\right) \qquad \forall p > 0.$$

We then relax the previous inequality by dropping the term $(1 - \bar{q}'_i)$ from the denominator of the left hand side and take the logarithm of both sides to get

$$\sum_{i:\bar{v}'_i \geq p} \ln\left(1 + \frac{\bar{v}'_i \bar{q}'_i}{p}\right) \leq \ln\left(\frac{p}{p-1}\right) \qquad \forall p > 0. \tag{17}$$

On the other hand, by invoking Lemma A.1 we can argue that [4]

$$\ln\left(1 + \frac{1}{\bar{v}_k}\right) + \sum_{i=2}^{k} \ln\left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k}\right) \leq \sum_{i=1}^{\min(k+j-2,n)} \ln\left(1 + \frac{\bar{v}'_i \bar{q}'_i}{\bar{v}_k}\right) \qquad \forall k \in \{2, \ldots, n\}. \tag{18}$$

---

[4] We invoke Lemma A.1 by setting $b = \frac{1}{\bar{v}_k}, a = \frac{\bar{v}_2 \bar{q}_2}{\bar{v}_k}, m = j$ and $z_i = \frac{\bar{v}'_i \bar{q}'_i}{\bar{v}_k}$.

Observe that for any given k the the right hand side of (18) is equal or less than the the left hand side of (17) for $p = \bar{v}_k$ which implies

$$\ln\left(1 + \frac{1}{\bar{v}_k}\right) + \sum_{i=2}^{k} \ln\left(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k}\right) \leq \ln\left(\frac{\bar{v}_k}{\bar{v}_k - 1}\right) \qquad \forall k \in \{2, \dots, n\}.$$

We can then further relax the above inequality by replacing the terms $\ln(1 + \frac{\bar{v}_i \bar{q}_i}{\bar{v}_k})$ with $\frac{1}{\bar{v}_k} \ln(1 + \bar{v}_i \bar{q}_i)$ and rearranging the terms to get the constraint (P4.1) which implies that $(\bar{\mathbf{v}}, \bar{\mathbf{q}})$ is feasible with respect to that constraint as well. ∎

Lemma 3.4. *The functions* $\mathcal{V}(p)$, $\mathcal{Q}(p)$, *and* $\mathcal{V}(p) - \mathcal{Q}(p)$ *are all decreasing in* p, *for* $p > 1$.

Proof: To prove $\mathcal{V}(p)$ is decreasing, we show its derivative is negative:

$$\mathcal{V}'(p) = \ln\left(1 + \frac{1}{p^2 - 1}\right) - \frac{2}{p^2 - 1} < \frac{1}{p^2 - 1} - \frac{2}{p^2 - 1} < 0.$$

The first inequality uses the fact that $\ln(1 + x) \leq x$. Similarly $\mathcal{Q}(p)$ is also decreasing because $\mathcal{Q}'(p) = \frac{1}{p}\mathcal{V}'(p) < 0$. Finally $\mathcal{V}(p) - \mathcal{Q}(p)$ is also decreasing because

$$\left(\mathcal{V}(p) - \mathcal{Q}(p)\right)' = \mathcal{V}'(p)\left(1 - \frac{1}{p}\right) < 0.$$

∎

Lemma 3.5. $\mathcal{V}(p) - \mathcal{V}(p') < \ln(\frac{p}{p-1}) - \ln(\frac{p'}{p'-1})$ *for any* $p' > p > 1$.

Proof: Define $G(p) = \mathcal{V}(p) - \ln(\frac{p}{p-1})$. Observe that proving the inequality in the statement of the lemma is equivalent to proving $G(p) < G(p')$ which we do by showing $G(p)$ has positive derivative and is therefore increasing.

$$G'(p) = \ln\left(1 + \frac{1}{p^2 - 1}\right) - \frac{1}{p^2 + p}.$$

We will show that $G'(p)$ is decreasing which then implies $G'(p) > 0$ because $\lim_{p \to \infty} G'(p) = 0$. Therefore we only need to show that $G''(p) < 0$.

$$G''(p) = \frac{-(3p + 1)}{(p-1)p^2(p+1)^2} < 0.$$

∎

Lemma A.1. *Consider any* $a, b, z_1, \dots, z_m \geq 0$ *such that* $a + b = \sum_{i=1}^{m} z_i$ *and* $a \leq z_m \leq b$. *Then*

$$\ln(1 + b) + \ln(1 + a) \leq \sum_{i=1}^{m} \ln(1 + z_i).$$

Proof: We can re-write the equation as

$$\ln\left((1 + a)(1 + b)\right) \leq \ln \prod_{i=1}^{m}(1 + z_i) \tag{19}$$

Observe that

$$\begin{aligned}
\prod_{i=1}^{m}(1 + z_i) &\geq 1 + \sum_{i=1}^{m} z_i + z_m\left(\sum_{i=1}^{m-1} z_i\right) \\
&= \left(1 + \sum_{i=1}^{m-1} z_i\right)(1 + z_m) \\
&\geq (1 + a)(1 + b),
\end{aligned}$$

where the first inequality follows by eliminating some terms from the expansion of $\prod_{i=1}^{m}(1 + z_i)$, and the second inequality from the assumption that $(1 + a) \le (1 + z_m)$ and $(1 + b) \ge (1 + \sum_{i=1}^{m-1} z_i)$. ∎