Replica Analysis of Large-system CDMA

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Abstract — **We present some new results on large-system CDMA obtained through the replica method developed in statistical physics. We find the spectral efficiency of randomly spread CDMA subject to Gaussian noise and flat fading in the large-system limit under arbitrary input distributions. Both joint decoding and single-user decoding are considered. In the latter case, a conditional mean estimator is first applied to separate the users and it is found that the resulting single-user channel for every user is equivalent to a Gaussian channel. The multiuser efficiency of that Gaussian channel is the same for all users and satisfies a fixed-point equilibrium equation. The additive decomposition by Shamai-Verdú of optimum capacity in terms of single-user capacity is shown to hold for arbitrary input distributions.**

I. Introduction

In the context of randomly spread code-division multiple access (CDMA), the inherent loss in spectral efficiency due to non-orthogonal signaling was quantified in the large-system limit using random matrix theory by Verdú and Shamai [1]. The expression found in [1] also solved the capacity of single-user narrowband multiantenna channels as the number of antennas grows—a problem that was open since the pioneering work of Foschini and Telatar. The complexity of optimal joint decoding, necessary for achieving capacity, is often prohibitive in practice. A common strategy is to separate the users using a multiuser detector front end and then perform single-user decoding, which induces further penalty on spectral efficiency. Assuming capacity-achieving Gaussian inputs, [1] found the penalty incurred by linear detectors as a function of the multiuser efficiency of the linear front end, also using random matrix theory.

The success of random matrix theory in the analysis of the spectral efficiency hinges on the fact that the performance measures for a finite number of users K and spreading factor N can be written as explicit functions of the singular values of the spreading matrix [2], the empirical distributions of which converge to a known function in the large-system limit where both K and N tend to infinity with a fixed ratio. As an important consequence, the dependency of the performance measures on the spreading sequences diminishes with probability 1. In other words, the performance measures are self-averaging. Random matrix theory underlies many interesting asymptotic results. In particular, the multiuser efficiency of the MMSE detector was found by Tse and Hanly [3]; and the spectral efficiency of Gaussian CDMA channels subject to flat fading was found by Shamai and Verdú [4] in the form of an additive decomposition as the sum of the capacity of a linear front-end and a nonlinear gain that depends only on the multiuser efficiency.

All the above results on spectral efficiency assume Gaussian inputs which are optimum because the channel realization is tracked at the receiver. Much less success has been reported in the application of random matrix theory to the analysis of the spectral efficiency achievable by specific signal constellations such as QPSK and 16QAM.

The self-averaging property is nothing but a manifestation of a fundamental law of nature that the fluctuation of macroscopic properties of certain many-body systems vanishes in the thermodynamic limit when the number of interacting bodies becomes large. In CDMA, the self-averaging principle ensures a strong property that for almost all realizations of the noise process and the spreading sequences, certain macroscopic average over the posterior probability distribution converges to the same number in the large-system limit, which is its ensemble average over the distribution of the noise and the spreading sequences. In [5], Tanaka pioneered statistical physics concepts and methodologies in multiuser detection and obtained the minimum bit-error-rate with antipodal uncoded inputs. Guo and Verdú [6] further elucidated the relationship between statistical physics and CDMA and generalized Tanaka's results to the case of non-equal powers. Inspired by [5], Müller and Gerstacker [7] studied the channel capacity under separated detection and decoding, where an optimal multiuser detector with no knowledge of the error-control codes is applied to generate soft decision statistics for each user for single-user decoding and noticed that the additive decomposition of [4] also holds for binary inputs. Müller thus further conjectured the same formula to be valid regardless of input distribution [8].

In this work, we use the replica method to analyze CDMA under a general framework, namely, the distribution of the input symbols as well as the received energies are arbitrary. Assuming that a detector separates the users by outputting the mean value of each symbol given the received signals for subsequent independent single-user decoding, we find that for each user, the resulting single-user channel is equivalent to a degraded Gaussian channel. The ratio of effective energy to true energy in the degraded Gaussian channel, called the *multiuser efficiency*, is found to satisfy a fixed-point equilibrium equation (a generalization of the Tse-Hanly equation [3]), and is the same for all users. The spectral efficiencies, both under joint and separate decoding, are found in explicit expressions in the multiuser efficiency, the input distribution and the fading characteristics. As an immediate corollary, Müller's conjec-

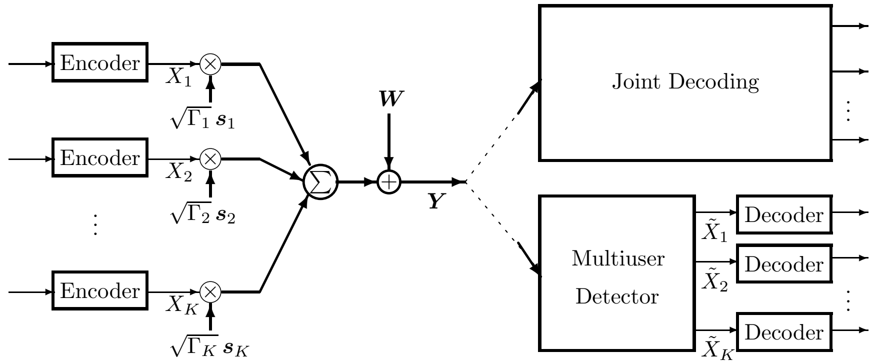

Figure 1: System model of CDMA with joint or separate decoding.

ture on the capacity loss is shown to be true, and we identify the loss as a Kullback-Leibler divergence between two Gaussian distributions.

The linear system in our study also models multiple-input multiple-output (MIMO) channels where the channel state is unknown at the transmitter. Our results can be used to evaluate the spectral efficiency of MIMO channels with high dimensionality under constellation constraints. An example is multiple antenna systems under homogeneous fading.

II. System Model and Summary of New Results

Consider the K-user CDMA system with spreading factor N depicted in Fig. 1. All symbols are independent identically distributed (i.i.d.) with distribution p_X, normalized so that $\mathsf{E}\left\{X^2\right\} = 1$. We use $\boldsymbol{X} = [X_1, \dots, X_K]^\top$ to denote a vector of input symbols from the K users. Let user k's spreading sequence be denoted by $\boldsymbol{s}_k = \frac{1}{\sqrt{N}}[s_{1k}, s_{2k}, \dots, s_{Nk}]^\top$, and the $N \times K$ spreading matrix denoted by $\boldsymbol{S} = [\boldsymbol{s}_1, \dots, \boldsymbol{s}_K]$, where the s_{nk}'s are i.i.d. random variables with zero mean, unit variance and finite higher-order moments. Assuming symbol-synchronous transmission, we have the following memoryless multiple-access channel:

$$\boldsymbol{Y} = \boldsymbol{S}\boldsymbol{A}\boldsymbol{X} + \boldsymbol{W} \tag{1}$$

where $\boldsymbol{W}$ is a vector that contains i.i.d. zero-mean unit variance Gaussian random variables, and $\boldsymbol{A} = \mathrm{diag}(\sqrt{\Gamma_1}, \dots, \sqrt{\Gamma_K})$ where Γ_k is the signal-to-interference ratio (SIR) of user k. We assume that the SIR of all users are known deterministic numbers, and as $K \to \infty$, its empirical cumulative distributions converge to a known distribution P_Γ, hereafter referred to as the *SIR distribution*. This SIR distribution captures the overall effect of the transmit energies, the noise level, and the fading characteristics of the channel.

The total capacity of the CDMA channel subject to a certain input distribution is the mutual information between the transmitted symbols $\boldsymbol{X}$ and the received signal $\boldsymbol{Y}$. There exists an error-correcting code of any rate no larger than the capacity such that by jointly decoding all users as depicted by the upper right block in Fig. 1, the original information can be recovered arbitrarily reliably.

Practically, we often break the process into multiuser detection followed by separate decoding, as shown by the lower right blocks in Fig. 1. A multiuser detector outputs an estimate of the transmitted symbols without knowledge of the error-control codes used by the encoder. Each decoder only takes the decision statistic for a single user of interest for decoding without awareness of the existence of any other users. We can regard the CDMA channel and the multiuser detector together as a superchannel, which is equivalent to K separate single-user channels. By the data processing inequality, the capacity of the superchannel, which is the sum of the single-user channel capacities, is less than the capacity of the original CDMA channel. In this work, we study a particular type of detector, namely, the *conditional mean estimator*, which outputs the mean value of the symbols conditioned on the received signal and the spreading sequences.

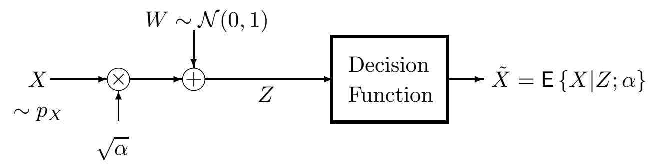

Figure 2: Equivalent single-user channel.

Clearly, the spectral efficiency under either joint or single-user (separate) decoding is dependent on the spreading sequences. However, if we consider the large-system asymptote, every performance measure we are interested in converges for almost all choices of the spreading sequences. Consequently, we can describe the multiuser efficiency and the spectral efficiency of the channel and the superchannel using merely macroscopic parameters without worrying about the instantaneous spreading sequences. In the following, we give the limit of the multiuser efficiency and the spectral efficiency when both K and N tend to infinity but with K/N converging to a positive number β. Besides β, we only assume that the input distribution p_X and the SIR distribution P_Γ are known.

Given a scalar $\alpha > 0$, consider a Gaussian channel as depicted in Fig. 2:

$$Z = \sqrt{\alpha}\, X + W \tag{2}$$

where W is a unit Gaussian random variable independent of X. We assume that the input X takes the distribution p_X.

Given that Z is received, we can define an estimate of X as

$$\tilde{X} = \mathsf{E}\{X|Z;\alpha\} \tag{3}$$

which is the mean value of X conditioned on Z. $\tilde{X}$ is the function of Z that minimizes the mean square error:

$$\mathcal{E}(\alpha) = \mathsf{E}\left\{ \left(\tilde{X} - X\right)^2 \middle| \alpha \right\}. \tag{4}$$

Theorem 1 *In the large-system limit, the distribution of the conditional mean estimator output $\tilde{X}_k$ of channel (1) conditioned on $X_k = x$ being transmitted is the same as that of the conditional mean estimate $\tilde{X}$ of channel (2) conditioned on $X = x$ being transmitted with $\alpha = \eta\Gamma_k$, where the multiuser efficiency η is the solution to the fixed-point equation:*

$$\eta + \eta \cdot \beta \cdot \mathsf{E}\{\Gamma \cdot \mathcal{E}(\eta\Gamma)\} = 1 \tag{5}$$

where the expectation is taken over the SIR distribution P_Γ.

Theorem 2 *In the large-system limit, the channel capacity under the conditional mean estimator and single-user decoding for a user with input distribution p_X and SIR Γ is equal to the input-output mutual information of the single-user Gaussian channel (2) with input distribution p_X and $\alpha = \eta\Gamma$ where η is the multiuser efficiency given by Theorem 1. In case of multiple solutions to (5), η is chosen as the smallest one.*

Theorem 3 *The gain of joint decoding over separate decoding in the large-system CDMA spectral efficiency is*

$$\frac{\eta - 1}{2}\log e - \frac{1}{2}\log\eta = \mathsf{D}\left(\mathcal{N}(0,\eta)\,||\,\mathcal{N}(0,1)\right). \tag{6}$$

The conditional mean estimator plays an important role in quantifying CDMA spectral efficiency. Theorem 1 reveals that each single-user channel resulting from applying the conditional mean estimator to a multiuser channel is equivalent to a degraded Gaussian channel as depicted in Fig. 2. The multiuser efficiency is a unique number in $[0,1]$ associated with CDMA as a solution to a fixed-point equation. The effective energy in every degraded channel is the input energy times the same multiuser efficiency. The single-user channel capacity is simply the mutual information across the degraded Gaussian channel under the input distribution p_X, as concluded in Theorem 2. The total spectral efficiency under joint decoding is closely related to that under separate decoding. The gain identified in Theorem 3 coincides with the expression found originally in [1] in the case of Gaussian inputs.

Our results on large CDMA systems allow a simple interpretation: The performance averaged over spreading sequences is equivalent to an equilibrium of the multiuser interference game. This can be best illustrated by introducing a canonical interference canceller as shown in Fig. 3. Suppose that the conditional mean estimates are available for all users but user 1. A decision statistic for user 1 is generated by first subtracting the reconstructed interferences using the estimates

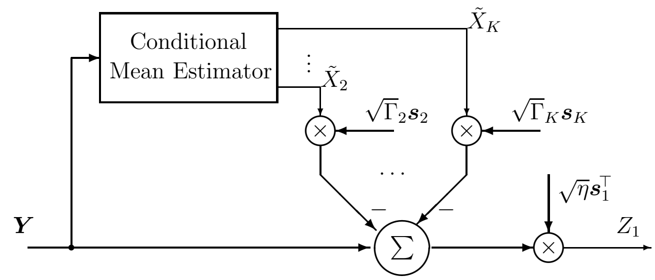

Figure 3: A canonical interference canceller equivalent to the conditional mean estimator.

and then matched filtering with respect to user 1's spreading sequence,

$$\frac{Z_1}{\sqrt{\eta}} = \sqrt{\Gamma_1}\,X_1 + \sum_{k=2}^{K} \boldsymbol{s}_1^\top \boldsymbol{s}_k \sqrt{\Gamma_k}\,(X_k - \tilde{X}_k) + W_1 \tag{7}$$

where W_1 is a unit Gaussian random variable. The variance of the multiple access interference, assuming the estimation errors are uncorrelated, is a weighted sum of $\mathcal{E}(\eta\Gamma_k)$. The resulting SIR for user 1 is therefore

$$\frac{\Gamma_1}{1 + \beta \cdot \mathsf{E}\{\Gamma \cdot \mathcal{E}(\eta\Gamma)\}}. \tag{8}$$

By the fixed-point equation in Theorem 1, (8) is equal to $\eta\Gamma_1$, which is exactly the same as that of a conditional mean estimator. We have thus shown that the conditional mean estimate for one user has the same quality as the output of an interference canceller using the conditional mean estimates of all other users. The multiuser efficiency is such that an equilibrium is achieved, so that every user enjoys the same efficiency; otherwise the users with worse efficiency may benefit from users with better efficiency until an equilibrium is reached.

III. Replica Analysis

The characteristic of the Gaussian CDMA channel with flat fading can be described as

$$p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{S}) = (2\pi)^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{x}\|^2\right]. \tag{9}$$

The posterior distribution can be obtained from the prior and the conditional distribution through the Bayes formula

$$\begin{aligned} & p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{S}) \\ =& Z^{-1}(\boldsymbol{y},\boldsymbol{S}) \cdot p_{\boldsymbol{X}}(\boldsymbol{x}) \cdot \exp\left[-\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{x}\|^2\right] \end{aligned} \tag{10}$$

where

$$Z(\boldsymbol{y},\boldsymbol{S}) = (2\pi)^{\frac{N}{2}} \cdot p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{S}) \tag{11}$$

is a normalizing coefficient, conveniently referred to as the *partition function*.

A. Joint Decoding

Since the input distribution is fixed, the total capacity under joint decoding is

$$I(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{S}) = \mathsf{E}\left\{ \log \frac{p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S})}{p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})} \,\middle|\, \boldsymbol{S} \right\}. \tag{12}$$

The spectral efficiency is defined as the total capacity divided by the number of chips per symbol interval. Therefore, by (11)–(12) and noticing that $p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}$ is a Gaussian density, the spectral efficiency of joint decoding is (in nats)

$$\mathsf{C}(\boldsymbol{S}) = \beta \cdot \mathsf{E}\left\{ -\frac{1}{K}\log Z(\boldsymbol{Y},\boldsymbol{S}) \middle| \boldsymbol{S} \right\} - \frac{1}{2}. \tag{13}$$

In statistical physics, the term inside the expectation in (13) is known as the free energy. As a macroscopic property, the free energy converges with probability 1 to its expectation in the large-system limit, which is denoted by $\mathcal{F}$,

$$\mathcal{F} = -\lim_{K\to\infty} \mathsf{E}\left\{ \frac{1}{K}\log Z(\boldsymbol{Y},\boldsymbol{S}) \right\}. \tag{14}$$

Therefore, the spectral efficiency converges almost surely to

$$\mathsf{C} = \beta\mathcal{F} - \frac{1}{2}. \tag{15}$$

The expectation of the logarithm in (14) is an open problem, for which the replica method is applied:

$$\begin{aligned}\mathcal{F} &= -\lim_{K\to\infty}\frac{1}{K}\lim_{u\to 0}\frac{\partial}{\partial u}\log \mathsf{E}\left\{Z^u(\boldsymbol{Y},\boldsymbol{S})\right\} &(16)\\ &= -\lim_{u\to 0}\frac{\partial}{\partial u}\lim_{K\to\infty}\frac{1}{K}\log \mathsf{E}\left\{Z^u(\boldsymbol{Y},\boldsymbol{S})\right\}. &(17)\end{aligned}$$

For an arbitrary integer u, we introduce u replicas of the CDMA system with the same received signal $\boldsymbol{Y}$ and spreading matrix $\boldsymbol{S}$. The partition function of the replicated system is

$$Z^u(\boldsymbol{y},\boldsymbol{S}) = \mathsf{E}\left\{ \prod_{a=1}^{u} \exp\left[-\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{X}_a\|^2 \right] \,\middle|\, \boldsymbol{S} \right\} \tag{18}$$

where the expectation is taken over the the i.i.d. replicated symbols, $\{X_{ak} | a = 1,\ldots,u,\ k = 1,\ldots,K\}$, conditioned on $\boldsymbol{S}$. We can henceforth evaluate

$$-\lim_{K\to\infty}\frac{1}{K}\log \mathsf{E}\left\{Z^u(\boldsymbol{Y},\boldsymbol{S})\right\} \tag{19}$$

as a function of the integer u. The *replica method* assumes that the resulting expression is also valid for an arbitrary real number u and find the derivative at $u = 0$ as the free energy. The replica method was invented in the context of spin glasses [9]. There are intensive ongoing efforts in the mathematics and physics community to find a rigorous proof for the replica method.

B. Separate Decoding

We assume that to separate the users we employ a conditional mean estimator:

$$\langle \boldsymbol{X} \rangle \stackrel{\triangle}{=} \mathsf{E}\left\{ \boldsymbol{X} \mid \boldsymbol{Y},\boldsymbol{S} \right\} \tag{20}$$

where the operator $\langle\cdot\rangle$ gives the expectation taken over the posterior probability distribution $p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$. The spectral efficiency is the mean value of the single-user channel capacities

$$I(X_k; \langle X_k \rangle \mid \boldsymbol{S}). \tag{21}$$

For the purpose of calculating (21), we study the joint distribution of X_k and $\langle X_k\rangle$ through their moments

$$\mathsf{E}\left\{ X_k^j \cdot \langle X_k\rangle^i \,\middle|\, \boldsymbol{S} \right\} \quad i,j = 0,1,\ldots \tag{22}$$

It turns out that the moments are macroscopic quantities which converge with probability 1 in the large-system limit to

$$\lim_{K\to\infty} \mathsf{E}\left\{ X_k^j \cdot \langle X_k\rangle^i \right\}. \tag{23}$$

Again, we use the replica method to evaluate (23). For notational convenience, we use $\boldsymbol{X}_0$ to denote the transmitted symbol vector. Upon each instance of $(\boldsymbol{Y},\boldsymbol{S})$, let $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_u$ be u independent random vectors induced from the posterior distribution $p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$. Thus $\boldsymbol{X}_0 \to (\boldsymbol{Y},\boldsymbol{S}) \to (\boldsymbol{X}_1,\ldots,\boldsymbol{X}_u)$ is a Markov chain. The moments in (23) are equivalent to

$$\lim_{K\to\infty} \mathsf{E}\left\{ X_{0k}^j \cdot \prod_{m=1}^{i} X_{mk} \right\} \tag{24}$$

which can be evaluated by working with a modified partition function akin to (18). The proof of Theorems 1–3 follows the above track.

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