Performance of Multicarrier CDMA in Frequency-Selective Fading Via Statistical Physics

Dongning Guo, *Member, IEEE*

***Abstract*—This correspondence extends previous work on direct-sequence code-division multiple access (DS-CDMA) to give a single-user characterization of multicarrier (MC) CDMA systems. Results on the error performance and information rate of a family of multiuser detectors for MC-CDMA are obtained using the replica method, which was originally developed in statistical physics. The central result is the "decoupling principle," namely, an MC-CDMA channel with frequency-selective fading followed by a generic multiuser detection front end can be decoupled into a bank of single-user fading channels in the large-system limit. Thus, conditioned on one's fading coefficients, each user essentially experiences an equivalent single-user Gaussian channel with a degradation in the signal-to-noise ratio (SNR) in lieu of multiaccess interference. A set of joint equations is identified, which determines the degradation for each user, known as the multiuser efficiency. The spectral efficiencies under both optimal joint decoding and separate single-user decoding following multiuser detection are obtained analytically. The result applies to arbitrary input distribution and SNRs, and to optimal multiuser detection as well as various suboptimal schemes.**

***Index Terms*—Channel capacity, multicarrier code-division multiple access (MC-CDMA), multiuser detection, replica method, spectral efficiency.**

I. Introduction

Multicarrier (MC) modulation is a highly attractive technique in both wired and wireless wideband communication due to its ease of equalization in presence of severe intersymbol interference [1]–[3]. Meanwhile, code-division multiple access (CDMA) is an appealing scheme in wireless multiuser applications due to its efficiency, ease of resource allocation, and many other advantages. A marriage of multicarrier and CDMA techniques provides the benefits of both and has great potential for serving as the air-interface for future high-rate mobile communication systems. Several multiaccess schemes that combine multicarrier techniques and CDMA have been proposed (e.g., [4]–[7]),[1] and a detailed discussion of various models is found in [8].

This work focuses on multicarrier CDMA (MC-CDMA), in which spreading is carried out in the frequency domain in contrast to direct-sequence (DS) CDMA [9]. Analytical results for MC-CDMA are mostly limited to binary inputs and linear detection (e.g., [10]–[14]) and numerical results are abundant. The channel capacity of MC-CDMA (with capacity-achieving Gaussian inputs) is studied in [15]. Statistical physics techniques were first applied to study the minimum probability of error under individually optimal detection in [16], where binary inputs and frequency-selective fading are assumed.

Following the introduction of statistical physics techniques to the treatment of DS-CDMA systems by Tanaka in [17], [18] developed a uniform framework to study flat-fading DS-CDMA with arbitrary input and a family of multiuser detectors in the large-system limit, where the number of users and the spreading factor both tend to infinity with a fixed ratio. The key result is the characterization of the "subchannel" between an individual user's input and the corresponding multiuser detection output. It is found that this subchannel is equivalent to a single-user Gaussian channel followed by a (nonlinear) one-to-one mapping, where the collective effect of the interference from other users is a degradation in the signal-to-noise ratio (SNR) of the equivalent channel. In other words, the DS-CDMA channel with a generic detection front end can essentially be decoupled into a bank of Gaussian single-user channels, the quality of which is determined by the SNR degradation, known as the multiuser efficiency. The result, called the "decoupling principle" in [18], holds for a family of detectors under the umbrella of posterior mean estimation, which includes but is not limited to the matched filter, the decorrelator, the linear minimum mean-square error (MMSE) detector, as well as the jointly optimal and individually optimal detectors.

Since MC-CDMA is the time–frequency dual of DS-CDMA, the results in [18] directly apply to MC-CDMA systems in the same setting: arbitrary inputs, generic multiuser detection, and flat fading.

What is new in this work is the performance of MC-CDMA under frequency-selective fading, i.e., the case where the subcarriers are subject to different fading coefficients, which is one of the major challenges in practical wideband mobile communications. A new decoupling principle for MC-CDMA is obtained: From an individual user's viewpoint, conditioned on this user's own fading coefficients, the multiaccess channel reduces to an equivalent single-user fading channel with enhanced noise. Consequently, in case of fast fading, the ergodic information rate that can be supported for a particular user if separately decoded, i.e., the average mutual information between the input and the detection output of the user, can be quantified as the input–output mutual information of the equivalent single-user Gaussian channel averaged over the fading coefficients. The total information rate, if all users are jointly decoded, is larger than the sum rate under separate decoding, while the gain due to joint decoding admits a simple expression in the multiuser efficiency. In case of slow fading, the outage probability can also be easily quantified. The results apply to arbitrary inputs and generic multiuser detection.

The decoupling results in this correspondence are obtained using the *replica method*, which was originally developed in statistical physics to study disordered systems [19]. The replica theory has proven to be useful in a variety of information processing problems, such as neural networks, error-control coding, and image processing, to name a few [20]. Indeed, many-user communication systems also find their counterpart as thermodynamic systems in physics, thus, they are amenable to the same line of analysis. Tanaka [17] was the first to use replicas to calculate the minimum probability of error of CDMA under optimal multiuser detection. The results have since been generalized to arbitrary input and generic multiuser detection [18], [21]. In fact, statistical physics methodologies have been applied not only to the analysis of multiuser detection with or without coding [16], [22], but also to the design of computationally feasible detection schemes [23]–[25].

To facilitate the use of statistical physics methodologies, some assumptions on the fading process are made in this correspondence. Although a rigorous justification for the replica method is still an open problem despite some recent progress [26], numerous instances suggest that the replica analysis provides an excellent approximation if not an exact result. In this correspondence, numerical results show that the decoupling principle can be quite accurate for systems with as few as eight users. This significantly simplifies the performance evaluation of multiuser systems and serves as a powerful interface for optimization on other design issues, such as signaling and power control (see e.g., [27]).

The remainder of this correspondence is organized as follows. Section II describes the MC-CDMA model. The performance of optimal

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The author is with the Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60208 USA (e-mail: dGuo@Northwestern.edu).
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[1]In [4] and [5] the scheme is called *spread-time CDMA*.

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and suboptimal detection is discussed in Sections III and IV, respectively. Numerical examples are presented in Section V before conclusions are drawn in Section VI. A sketch of the replica analysis used to obtain the results is relegated to the Appendix.

II. Multicarrier CDMA

Consider an MC-CDMA system with L equally spaced orthogonal subcarriers shared by K synchronous users. Let $\boldsymbol{X} = [X_1, \dots, X_K]^\top$ denote the vector of independent and identically distributed (i.i.d.) input symbols with distribution p_X from the K users in one symbol interval.[2] Denote the frequency-domain spreading sequence of user k by $\boldsymbol{s}_k = \frac{1}{\sqrt{L}}[S_{1k}, S_{2k}, \dots, S_{Lk}]^\top$, where S_{nk} are i.i.d. with zero mean, unit variance, and finite moments. The $L \times K$ channel state matrix is denoted by $\boldsymbol{S} = [\boldsymbol{s}_1, \dots, \boldsymbol{s}_K]$. The baseband signal transmitted by user k in the time domain is $\boldsymbol{F}\boldsymbol{s}_k X_k$ where $\boldsymbol{F}$ is the $L \times L$ inverse discrete Fourier transform (IDFT) matrix, i.e., $F_{ln} = \exp[j2\pi(n-1)(l-1)/L]$. For ease of equalization, a cyclic prefix of sufficient length is prepended to the baseband signal. The receiver performs DFT to recover signals in the frequency domain.

Due to frequency-selective fading, the L subcarriers for each user may be subject to different fading coefficients. Let snr_k capture the long-term average SNR per symbol for user k. Let H_{lk} denote the instantaneous complex-valued fading coefficient at subcarrier l for user k. The signature waveform for user k in the frequency-domain is described by

$$\tilde{\boldsymbol{s}}_k = \sqrt{\frac{\mathsf{snr}_k}{L}}[H_{1k}S_{1k}, H_{2k}S_{2k} \dots, H_{Lk}S_{Lk}]^\top.$$

Thanks to the cyclic prefix, circular convolution in the time domain corresponds to the Hadamard product in the frequency domain. Let $\circ$ denote element-wise (aka. Hadamard) product. The received signal vector is described in the frequency domain by [7]:

$$\begin{aligned}\boldsymbol{Y} &= \sum_{k=1}^{K} \tilde{\boldsymbol{s}}_k X_k + \boldsymbol{N} \\ &= (\boldsymbol{H} \circ \boldsymbol{S})\boldsymbol{A}\boldsymbol{X} + \boldsymbol{N} \end{aligned} \tag{1}$$

where $\boldsymbol{A} = \mathrm{diag}\{\sqrt{\mathsf{snr}_1}, \dots, \sqrt{\mathsf{snr}_K}\}$ and the noise vector $\boldsymbol{N}$ consists of i.i.d. unit circularly symmetric complex Gaussian entries (denoted by $\mathcal{CN}(0,1)$).

The use of the cyclic prefix induces loss in both energy and bandwidth, which is inversely proportional to the number of subcarriers and easy to quantify. Disregarding the cyclic prefix overhead, the only difference between model (1) and that of a flat-fading CDMA channel is that the spreading chips (subcarriers) for each user may be subject to different fading.

The uplink and downlink are distinguished in the analysis due to their different fading statistics. In the downlink, all users experience the same fading so that the columns of $\boldsymbol{H}$ are identical, while in the uplink, the fading processes for different users are independent due to geographical separation, i.e., the columns of $\boldsymbol{H}$ are independent in the uplink. In order to facilitate the large-system analysis, we assume a "block-fading" model in the subcarriers (as opposed to block fading in time) in both the uplink and the downlink. That is, the L subcarriers are divided into M segments of consecutive subcarriers, where each segment of L/M subcarriers experiences the same fading coefficient, while the fading processes of the M segments are independent. Precisely, $H_{lk} = H_{l'k}$ for every $m \in \{1, \dots, M\}$, and $l, l' \in G_m$ where

$$G_m = \{(m-1)L/M + 1, \dots, mL/M\}.$$

[2]For convenience, p_X is used to denote the probability density or mass function. The results holds for general input distributions and do not depend on the existence of a probability density or mass function.

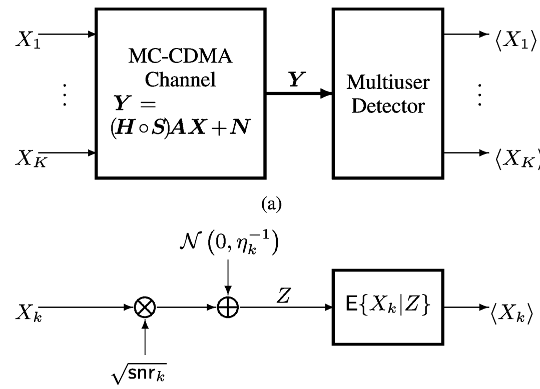

Fig. 1. (a) A multiuser system and (b) its single-user equivalence for user k.

Let W_m, $m = 1, \dots, M$, be independent random variables that describe the fading characteristics of the M groups of subcarriers in terms of energy. Then for each k, $|H_{lk}|^2$ takes the same distribution as W_m for all $l \in G_m$.

In the analysis, we will keep M finite and let the number of users K and the spreading factor L both tend to infinity with the system load K/L converging to a given number $\beta \in (0, \infty)$. This asymptote is referred to as the large-system limit. The above assumption on the large-system fading characteristics is different from that in [15], where a two-dimensional channel profile function is defined to study the Shannon capacity of MC-CDMA in the large-system limit. In particular, [15] assumes fixed coherence bandwidth such that correlation is limited to a fixed number of neighboring subcarriers as the total number grows. As far as the performance of finite-size systems is concerned, the subcarrier block fading assumption is equally valid as the one in [15], since after all, the goal is to obtain a large-system result that is well representative of systems of given dimensions (K, L, and M). Indeed, numerical results are shown in Section V to support the approach taken by this correspondence.

Model (1) describes a snapshot of the input–output relationship of the MC-CDMA channel during one symbol interval. Assume that the spreading matrix and the SNRs are known and that the variation of the fading coefficients can be followed by the receiver, i.e., $\tilde{\boldsymbol{S}} = (\boldsymbol{H} \circ \boldsymbol{S})\boldsymbol{A}$ is known (or can be estimated accurately). Maximum information capacity is achieved by optimal joint decoding, which is prohibitively complex in practice. A computationally more feasible scheme is to perform multiuser detection as depicted in Fig. 1(a) before independent decoding based on individual user's detection outputs. By adopting this separate decoding approach, the multiuser channel together with the detection front end is viewed as a bank of coupled subchannels, one for each user.

In this work, the replica method is applied to study the large-system limit of the error performance and the information rate, where we obtain a single-user characterization of the subchannels. The results for optimal and suboptimal detection are discussed in Sections III and IV respectively.

III. Optimal Detection

Consider first optimal detection for the multiuser system in mean-square sense, which produces the posterior mean estimate

$$\langle X_k \rangle = \mathsf{E}\left\{X_k \mid \boldsymbol{Y}, \tilde{\boldsymbol{S}}\right\}. \tag{2}$$

Here, the operator $\langle\cdot\rangle$ denotes the posterior mean with respect to $p_{\boldsymbol{X}|\boldsymbol{Y},\tilde{\boldsymbol{S}}}$ (also known as the conditional mean). In this correspondence, (2) is also referred to as the individually optimal detector. Note that in case of binary transmission, hard decision based on the posterior mean (2) gives the minimum probability of error. The optimal detector (2) is in general nonlinear, while in case the input distribution $p_{\boldsymbol{X}}$ is Gaussian, it is exactly the linear MMSE detector. In general, computing the error performance of the optimal detector is a hard problem [9].

Using the replica method, the large-system performance of optimal detection has been obtained for DS-CDMA [17], [18], which is a special case of model (1) with $H_{lk} = 1$ for all l, k, where snr_k are i.i.d. with the SNR distribution p_{snr} which incorporates flat fading. It is found that such a CDMA channel, together with the multiuser detector front end (see Fig. 1(a)), can be decoupled into a bank of Gaussian single-user channels. As depicted in Fig. 1(b), the subchannel for user k is equivalent to a single-user Gaussian channel concatenated with a (nonlinear) mapping, which is the posterior mean estimator that corresponds to the single-user channel. The collective effect of the interfering users is equivalent to an enhancement of the Gaussian noise, or, in other words, a degradation in the SNR of the desired user. The degradation factor η, known as the multiuser efficiency, is the same for all users and satisfies the following fixed-point equation:

$$\eta^{-1} = 1 + \beta \mathsf{E}\left\{\mathsf{snr} \cdot \mathsf{mmse}(\eta\, \mathsf{snr})\right\} \tag{3}$$

where the expectation is over p_{snr}. Here, $\mathsf{mmse}(\gamma)$ denotes the MMSE in estimating the input of distribution p_X using the output of the single-user Gaussian channel with its SNR equal to γ:

$$\mathsf{mmse}(\gamma) = \mathsf{E}\left\{|X - \mathsf{E}\left\{X \mid \sqrt{\gamma}X + N\right\}|^2\right\}$$

where $X \sim p_X$ and $N \sim \mathcal{CN}(0,1)$. It is clear that the larger the MMSE (i.e., the detection error), the smaller the efficiency. The above result is referred to as the *decoupling principle*, since asymptotically, each user experiences an equivalent single-user channel with additive Gaussian noise independent of the interferers' data.

Formula (3) directly applies to MC-CDMA with flat fading. In this work, we modify the previous replica analysis for DS-CDMA [18] in order for a treatment of the general model (1) for MC-CDMA with frequency-selective fading. In its full manner, the replica analysis involves calculating the asymptotic free energy of the MC-CDMA system and joint moments of some replicas of the input symbols [18]. A sketch of the replica analysis that derives the results is relegated to the Appendix which highlights the differences from that for DS-CDMA in [18]. The following decoupling principle is obtained.

Proposition 1: The MC-CDMA channel described by (1) with the optimal multiuser detector front end given by (2) can be decoupled into a bank of single-user fading subchannels.

1) In the downlink, conditioned on that the power of the fading coefficients being $W_m = w_m$ for the subcarrier groups $m = 1, \dots, M$, the performance of user k is essentially equivalent to that of a single user observed in Gaussian noise with the same input and SNR equal to $\eta(\boldsymbol{w})\mathsf{snr}_k$, where the multiuser efficiency, as a function of $\boldsymbol{w} = [w_1, \dots, w_M]$, satisfies the following fixed-point equation:

$$\eta(\boldsymbol{w}) = \frac{1}{M}\sum_{m=1}^{M}\left[\frac{1}{w_m} + \beta\mathsf{E}\left\{\mathsf{snr}\cdot\mathsf{mmse}(\eta(\boldsymbol{w})\,\mathsf{snr})\right\}\right]^{-1}. \tag{4}$$

2) In the uplink, let the powers of the fading coefficients for user k in the subcarrier group m be w_{mk}. The decoupling principle holds, where the equivalent SNR for user k is $\eta(\boldsymbol{w})\mathsf{snr}_k$, where the multiuser efficiency, as a function of user k's fading coefficients $\boldsymbol{w}_k = [w_{1k}, \dots, w_{Mk}]$ satisfies the following fixed-point equation:

$$\eta(\boldsymbol{w}_k) = \sum_{m=1}^{M}\frac{w_{mk}}{M}\left[1 + \beta\mathsf{E}\left\{\mathsf{snr}\,\bar{W}_m \mathsf{mmse}(\eta(\bar{\boldsymbol{W}})\mathsf{snr})\right\}\right]^{-1} \tag{5}$$

where the expectation is over p_{snr} as well as the fading characteristic $\bar{\boldsymbol{W}} = \left[\bar{W}_1, \dots, \bar{W}_M\right]$ where $\bar{W}_m$ is identically distributed as W_{mk} with k equally likely to be $1, \dots, K$ (i.e., a mixture of W_{mk}).

It is important to note that, in both the uplink and the downlink, the multiuser efficiency for an individual user is dependent on the instantaneous fading powers of the user and therefore time varying. Indeed, as a natural outcome of the fading subcarriers, the equivalent single-user channels are also subject to fading. The long-term average performance is determined by either an average over the fading coefficients or the outage probability depending on whether slow or fast fading is the case. Unlike under some other large-system settings, where the effect of frequency-selective fading is averaged out, our assumption leads to an equivalent single-user fading channel, which is particularly relevant to the performance of finite-size systems. By inspection, (4) and (5) are consistent with the result for flat fading (3) when $w_{lk} = 1$. Also, the downlink can be regarded as a special case of the uplink with $w_{mk} = w_m$ (and hence, $\bar{W}_m = w_m$) for all k.

Proposition 1 provides a good approximation for systems of moderate size and load, which becomes increasingly accurate as the system size increases. Note that the instantaneous multiuser efficiencies can be greater than 1 due to fading, but is upper-bounded by $(1/M)\sum_{m=1}^{M} w_m$.

The fixed point (4) for the downlink can, in general, be solved numerically for any given $\boldsymbol{w}$. Note that the efficiency can be understood as a weighted sum of some efficiencies associated with each group of subcarriers

$$\eta = \frac{1}{M}\sum_{m=1}^{M} w_m \eta_m$$

where for $m = 1, \dots, M$

$$\eta_m^{-1} = 1 + \beta w_m \mathsf{E}\left\{\mathsf{snr}\cdot\mathsf{mmse}\left(\frac{\mathsf{snr}}{M}\sum_{m=1}^{M} w_m\eta_m\right)\right\}. \tag{6}$$

The fixed point (5) for the uplink is harder to deal with directly because the expression for $\eta(\cdot)$ is self-referenced in the expectation. The equation can be easier to solve in the following equivalent form:

$$\eta_m^{-1} = 1 + \beta\mathsf{E}\left\{\mathsf{snr}\,\bar{W}_m\;\mathsf{mmse}\left(\frac{\mathsf{snr}}{M}\sum_{m=1}^{M}\bar{W}_m\eta_m\right)\right\} \tag{7}$$

for $m = 1, \dots, M$ where η_m are deterministic numbers that can be solved jointly from (7), and the efficiency for user k is

$$\eta^{(k)} = \frac{1}{M}\sum_{m=1}^{M} w_{mk}\eta_m. \tag{8}$$

It appears that the difference between the downlink result (6) and the uplink result (7) is whether the average is taken with respect to the fading coefficients of the interfering users.

In the special case of uplink, where all users' fading characteristics are homogeneous, i.e., for each m, W_{mk} are i.i.d. for all k, it follows from (5) that the multiuser efficiency for all users are the same. Nonetheless, the uplink differs from the downlink because w_{mk} and $w_{mk'}$ are not equal in general for $k \neq k'$. The set of fixed-point equation (7) collapses into the following and is much easier to solve:

$$\eta_1^{-1} = 1 + \beta\mathsf{E}\left\{\mathsf{snr}\;\bar{W}\;\mathsf{mmse}\left(\mathsf{snr}\bar{W}\eta_1\right)\right\}$$

where $\bar{W}$ takes the same distribution as $\frac{1}{M}\sum_{m=1}^{M}\bar{W}_m$. In fact, (7) is the same as the result for flat fading (3) only with the SNR distribution incorporating the homogeneous fading. The multiuser efficiency for user k is given by (8) with $\eta_m = \eta_1$ for all m.

The decoupling principle implies that the information rate for each user is the same as the equivalent single-user channel. The following corollary is relevant to both the downlink and the uplink.

Corollary 1: The mutual information between the input and detection output for user k conditioned on the fading characteristics converges in probability in the large-system limit to the mutual information of the equivalent single-user channel

$$I(X_k; \langle X_k \rangle | \boldsymbol{H}, \boldsymbol{S}) \to I\left(X_k; \sqrt{\eta(\boldsymbol{W}_k)\,\mathsf{snr}}X_k + N|\boldsymbol{W}_k\right) \tag{9}$$

where $\boldsymbol{W}_k = [W_{1k}, \ldots, W_{Mk}]$, $N \sim \mathcal{CN}(0,1)$, and the multiuser efficiency $\eta(\cdot)$ is determined by Proposition 1 in both the uplink and the downlink.

Since the optimal detection output for an individual user is not a sufficient statistic of the original received signal for detecting the desired user's signal, the overall mutual information under joint decoding (which is essentially the free energy of the multiuser system [18]) is greater than the sum of individual mutual informations. Indeed, the gain due to joint decoding is given by the following result.

Proposition 2: The downlink spectral efficiency under optimal joint decoding, i.e., the input–output mutual information of channel (1) multiplied by the load, is obtained as

$$\mathsf{C}_{\text{joint}} = \mathsf{E}\left\{\beta I\left(X; \sqrt{\eta(\boldsymbol{W})\mathsf{snr}}X + N|\boldsymbol{W}\right) + \frac{1}{M}\sum_{m=1}^{M}\left[\frac{\eta_m - 1}{W_m} - \log \eta_m\right]\right\}$$

where $\eta(\cdot)$ and η_m are determined by (4) and (6), respectively.

In the uplink, the spectral efficiency under optimal joint decoding is

$$\mathsf{C}_{\text{joint}} = \beta\mathsf{E}\left\{I\left(X; \sqrt{\eta(\bar{\boldsymbol{W}})\mathsf{snr}}X + N|\bar{\boldsymbol{W}}\right)\right\} + \frac{1}{M}\sum_{m=1}^{M}[\eta_m - 1 - \log \eta_m]$$

where $\eta(\cdot)$ and η_m are determined by (5) and (7), respectively.

We point out that under nonlinear multiuser detection, the fixed points (4) and (5) may have multiple sets of solutions. The multiuser efficiency that minimizes the joint detection spectral efficiency given by Proposition 2 (or, equivalently, the free energy) is the one that carriers the operational meaning of the SNR degradation [18].

IV. Suboptimal Detection

The preceding discussion is limited to optimal multiuser detection. In fact, the decoupling principle also applies to a broad family of suboptimal detectors. Following [18], a generic (suboptimal) multiuser detector is defined for model (1) as the optimal detector for a postulated model described by

$$\boldsymbol{Y} = (\boldsymbol{H} \circ \boldsymbol{S})\boldsymbol{A}\boldsymbol{X}' + \sigma\boldsymbol{N}$$

where $\boldsymbol{H}$, $\boldsymbol{S}$, $\boldsymbol{A}$, and $\boldsymbol{N}$ remain the same as in (1), while the components of the input $\boldsymbol{X}'$ are i.i.d. with distribution q_X, and σ is the postulated noise level. The generic detector is suboptimal for the actual model (1) because of mismatch in the input distribution (postulated q_X versus actual p_X) and noise level (σ versus 1). The generic detector is nonetheless a posterior mean estimator

$$\langle X_k \rangle_q = \mathsf{E}_q\left\{X_k \mid \boldsymbol{Y}, \tilde{\boldsymbol{S}}\right\} \tag{10}$$

where the expectation $\mathsf{E}_q\{\cdot\}$ is taken over the posterior $q_{\boldsymbol{X}|\boldsymbol{Y},\tilde{\boldsymbol{S}}}$ associated with the postulated model. Note that $q_{\boldsymbol{X}|\boldsymbol{Y},\tilde{\boldsymbol{S}}}$ is determined by q_X and $q_{\boldsymbol{Y}|X,\tilde{\boldsymbol{S}}}$ using Bayes' formula. The multiuser channel and the generic detector are depicted by Fig. 1(a) with $\langle X_k \rangle$ replaced by $\langle X_k \rangle_q$, which is defined by (10). In view of (2), the subscript in (10) can be dropped if the postulated measure q coincides with the actual one p.

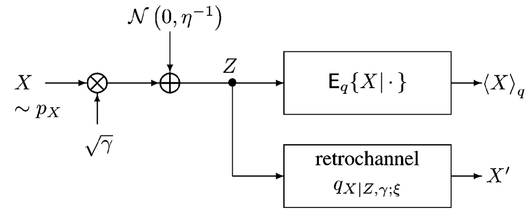

Fig. 2. The equivalent single-user Gaussian channel, posterior mean estimator, and retrochannel.

In general, postulating $q \neq p$ causes degradation in detection performance. Such a strategy may be either due to lack of knowledge of the true statistics p or a particular choice that corresponds to a certain suboptimal detector of interest. As shown in [18], by postulating an appropriate measure q, the generic detector can be particularized to well-known multiuser detectors. For example, if the postulated distribution for all input symbols is standard Gaussian, the posterior mean (10) is a linear estimate. Furthermore, if $\sigma = 1$, (10) gives exactly the soft output of the linear MMSE detector, whereas by sending $\sigma \to \infty$, it becomes equivalent to the single-user matched filter output. Alternatively, if q_X is identical to p_X, then $\sigma = 1$ corresponds to the individually optimal detector, while $\sigma \to 0$ corresponds to the jointly optimal detector [9], [18].

The generic representation (10), which is parameterized by (q_X, σ), enables a uniform treatment of a large family of multiuser detectors and thereby results in a simple universal characterization. Reference [18] studies flat-fading DS-CDMA with generic multiuser detection and shows that the decoupling principle still holds, while the multiuser efficiency η together with an auxiliary parameter ξ satisfies a pair of fixed-point equations.

The replica analysis in [18] can also be carried out for MC-CDMA with the generic front end (10) to obtain a general decoupling result. The analysis follows the sketch given in the Appendix with additional considerations of generic posterior mean estimation.

In order to state the decoupling result, we introduce two quantities derived from the single-user model in Fig. 2. Let X take distribution p_X. Conditioned on X, Z is Gaussian with mean $\sqrt{\gamma}X$ and variance η^{-1}. Let $q_{Z|X,\gamma;\xi}$ be the conditional distribution that describes the following postulated Gaussian channel:

$$Z = \sqrt{\gamma}X' + N/\sqrt{\xi}$$

where $N \sim \mathcal{CN}(0,1)$. Let $q_{X|Z,\gamma;\xi}$ be the posterior distribution induced by $q_{Z|X,\gamma;\xi}$ and the postulated input distribution q_X using Bayes' formula. In [18], the transformation described by $q_{X|Z,\gamma;\xi}$ (from Z to X') is called the retrochannel. The output of the conditional mean estimator is

$$\langle X \rangle_q = \mathsf{E}_q\{X \mid Z, \gamma; \xi\}$$

where the expectation is with respect to $q_{X|Z,\gamma;\xi}$. With the above conditional distributions defined, X–Z–X' is a Markov chain and their joint distribution is determined. In fact, we can think of the model depicted in Fig. 2 as a probabilistic machine driven by the input $X \sim p_X$ and the realization of the Gaussian noises.

Define the function $\mathcal{E}(\gamma; \eta, \xi)$ as the mean-square error achieved by the generic detector $\mathsf{E}_q\{X \mid Z, \gamma; \xi\}$

$$\mathcal{E}(\gamma; \eta, \xi) = \mathsf{E}\left\{\left|X - \mathsf{E}_q\{X \mid Z, \gamma; \xi\}\right|^2\right\}$$

where

$$Z = \sqrt{\gamma}X + N/\sqrt{\eta}.$$

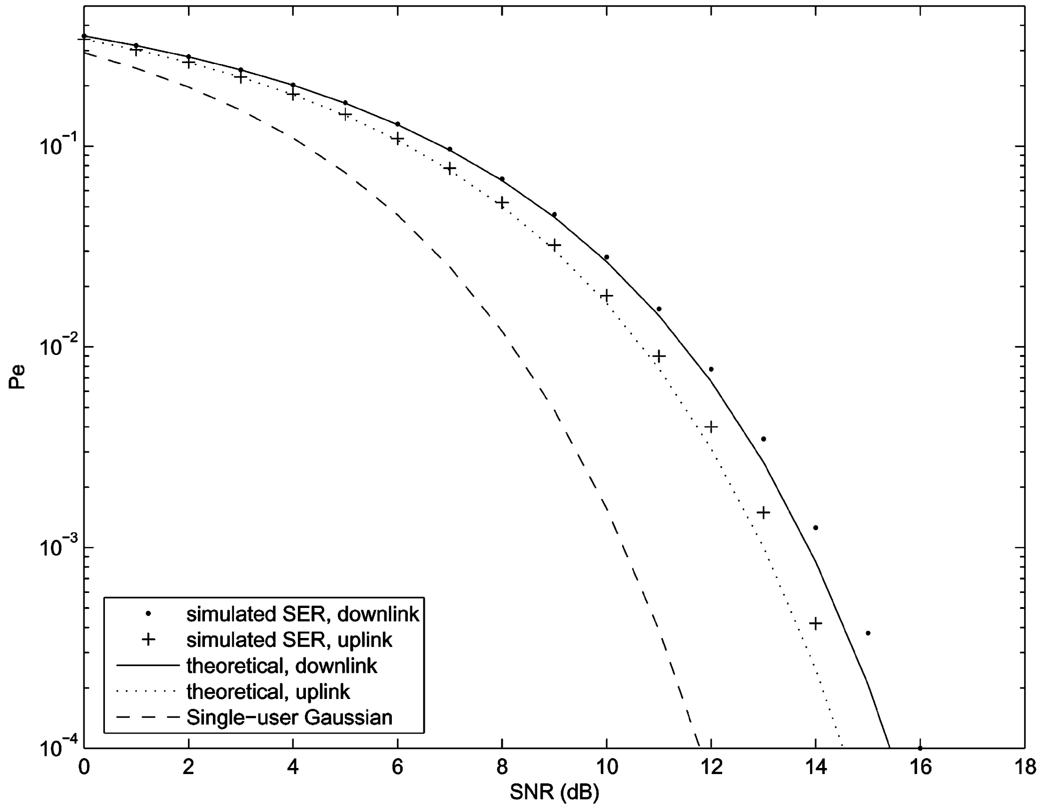

Fig. 3. MC-CDMA with linear MMSE detection. Simulated and theoretical SERs, $K = 32, L = 64$.

We also define a variance function $\mathcal{V}(\gamma; \eta, \xi)$ associated with the retrochannel, which is the expected value of the conditional variance of X' given Z

$$\mathcal{V}(\gamma; \eta, \xi) = \mathsf{E}\left\{\left|X' - \mathsf{E}_q\left\{X \mid Z, \gamma; \xi\right\}\right|^2\right\}.$$

It is found that, in the large-system limit, an equivalence of the subchannel for user k is depicted by Fig. 1(b) with the conditional mean estimator replaced by a generic one: $\mathsf{E}_q\left\{X_k \mid Z; \mathsf{snr}_k, \xi\right\}$. For user k, this generic (nonlinear) decision function is the conditional mean estimator for a Gaussian channel with an input distribution q_X and SNR equal to $\xi\ \mathsf{snr}_k$.

The decoupling principle for MC-CDMA with generic multiuser detection is given as the following.

Proposition 3: Consider a generic multiuser detector parameterized by (q_X, σ). The MC-CDMA channel followed by the generic detector can essentially be decoupled into a bank of single-user fading channels. The multiuser efficiency η and an auxiliary variable ξ satisfy the following joint equatons:

$$\xi(\boldsymbol{w}) = \sum_{m=1}^{M} \frac{w_m}{M} \frac{1}{\sigma^2 + \beta \mathsf{E}\left\{\bar{W}_m\ \mathsf{snr}\ \mathcal{V}\left(\mathsf{snr}; \eta(\bar{\boldsymbol{W}}), \xi(\bar{\boldsymbol{W}})\right)\right\}} \tag{11a}$$

$$\frac{\xi^2(\boldsymbol{w})}{\eta(\boldsymbol{w})} = \sum_{m=1}^{M} \frac{w_m}{M} \frac{1 + \beta \mathsf{E}\left\{\bar{W}_m\ \mathsf{snr}\ \mathcal{E}\left(\mathsf{snr}; \eta(\bar{\boldsymbol{W}}), \xi(\bar{\boldsymbol{W}})\right)\right\}}{\left[\sigma^2 + \beta \mathsf{E}\left\{\bar{W}_m\ \mathsf{snr}\ \mathcal{V}\left(\mathsf{snr}; \eta(\bar{\boldsymbol{W}}), \xi(\bar{\boldsymbol{W}})\right)\right\}\right]^2}. \tag{11b}$$

The result applies to both the downlink and the uplink. In the uplink, $\eta(\boldsymbol{w})$ gives the spectral efficiency of an individual user whose fading coefficients are given by $\boldsymbol{w}$, whereas $\bar{\boldsymbol{W}}$ stands for the random fading coefficients associated with the groups of subcarriers of all users (i.e., a mixture of the fading distribution of all users). In the downlink, $\bar{\boldsymbol{W}} = \boldsymbol{w}$ which is the fading coefficients for the M groups of subcarriers.

Proposition 3 not only encompasses Proposition 1 and previous error performance results as special cases (e.g., [15], [16]), but also gives for the first time the performance of generic detectors including the (nonlinear) optimal detection with arbitrary input distribution in MC-CDMA with frequency-selective fading.

The set of joint equations (11) that determines the multiuser efficiencies can be solved numerically given moderate system size. The computation can be made easier by symmetry if the users are further categorized into a few groups of the same or similar fading characteristics which share the same efficiencies. At any rate, the decoupling results significantly simplify the evaluation of the performance of MC-CDMA systems.

Similar to Corollary 1, the mutual information of the subchannel for each user is determined by the multiuser efficiency as given in the following.

Corollary 2: The mutual information between the input and detection output for user k converges in probability in the large-system limit to the mutual information of the equivalent single-user channel, i.e., (9) holds literally where $\eta(\boldsymbol{W}_k)$ is the multiuser efficiency of user k determined by Proposition 3.

V. Numerical Results

This section plots the error performance of MC-CDMA with frequency-selective fading. Assume that the gains of all the subcarriers are equally spaced, with the highest power subcarrier 10 dB stronger than that of the weakest. Both the optimal posterior mean estimator and the linear MMSE detector are considered. The symbol-error rate (SER) is plotted against the average SNR of all users, and compared to theoretical error performance obtained using formulas given in this correspondence.

Fig. 3 shows the uplink and downlink performance of a 32-user system with spreading factor 64. Quatenary phase-shift keying (QPSK)

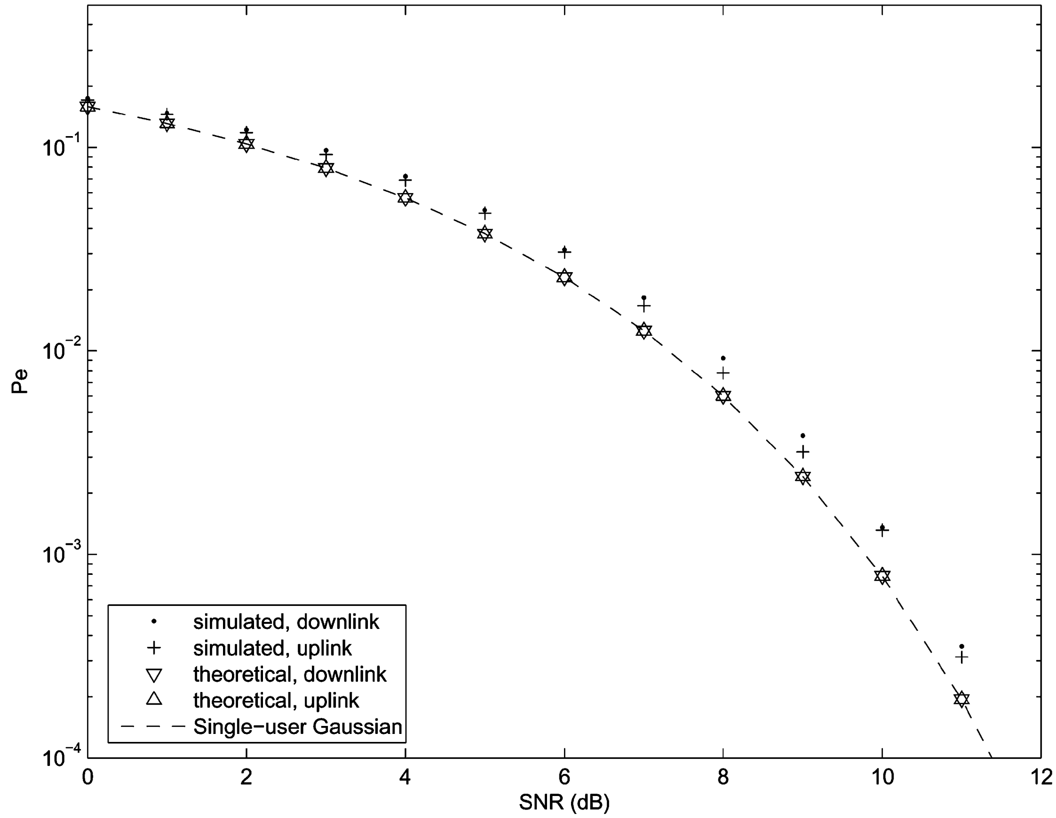

Fig. 4. MC-CDMA with individually optimal detection. Simulated and theoretical bit-error rates, $K = 16$, $L = 64$.

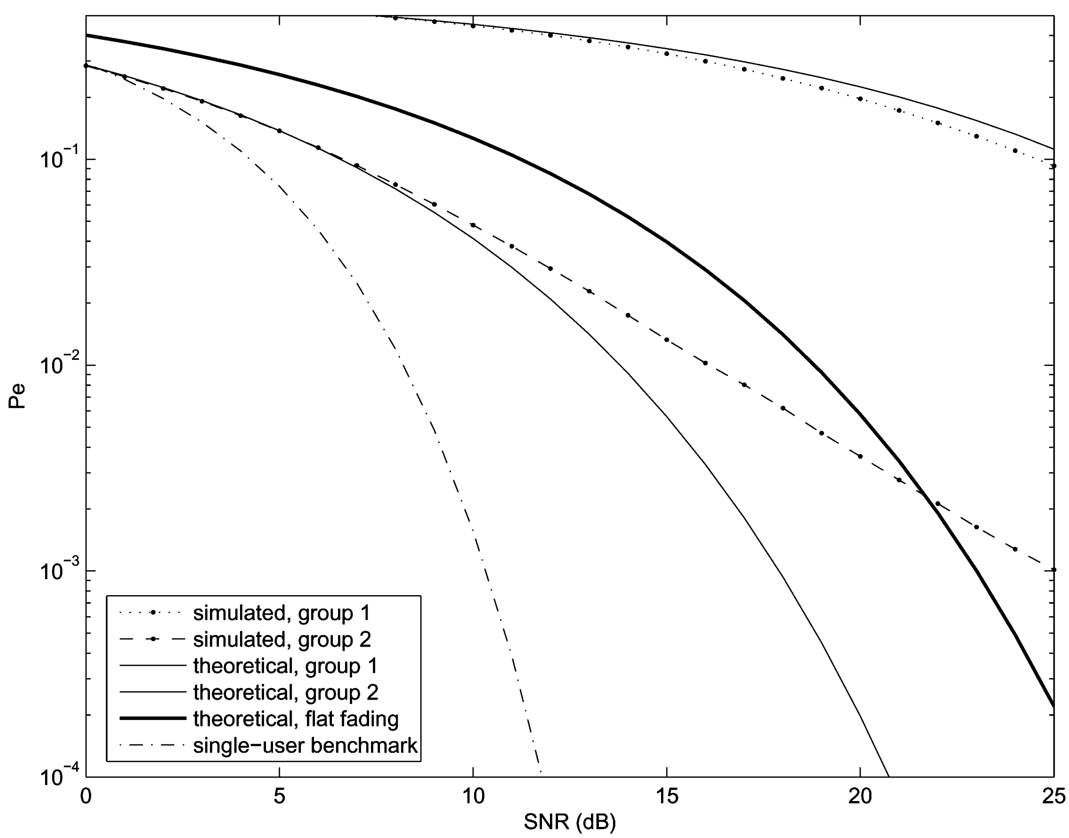

Fig. 5. MC-CDMA with linear MMSE detection. Simulated and theoretical SERs, $K = 8$, $L = 8$.

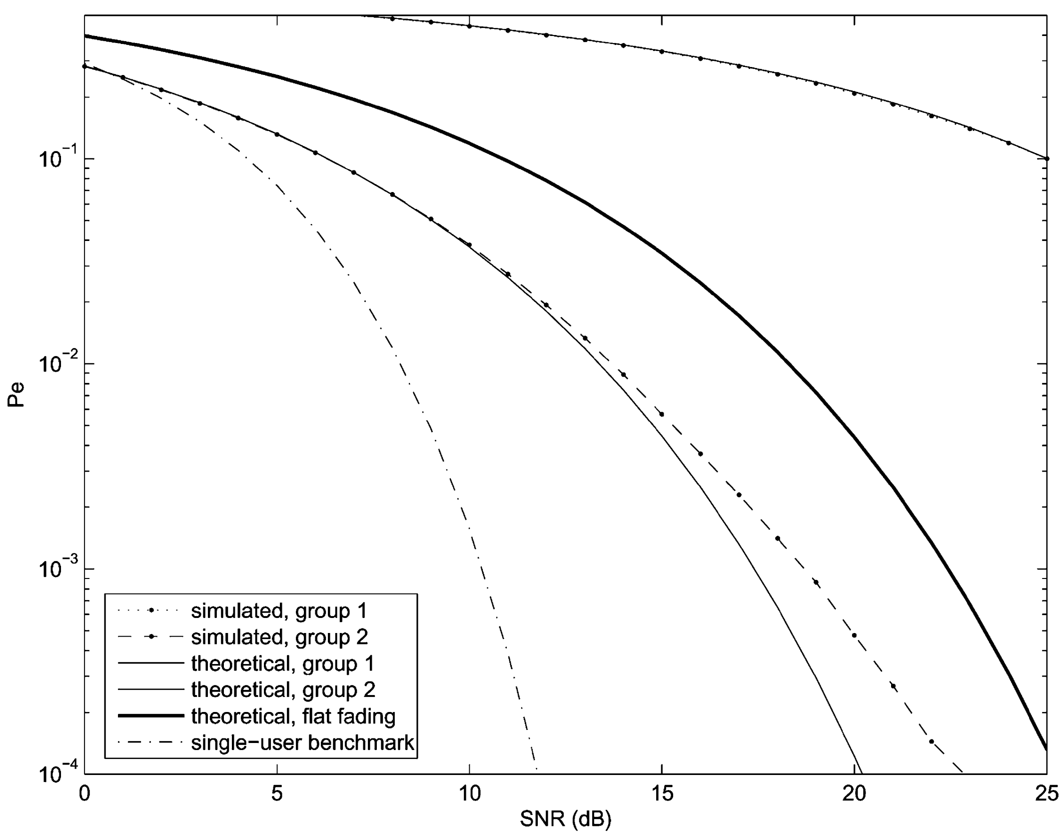

Fig. 6. MC-CDMA with linear MMSE detection. Simulated and theoretical SERs, $K = 64, L = 64$.

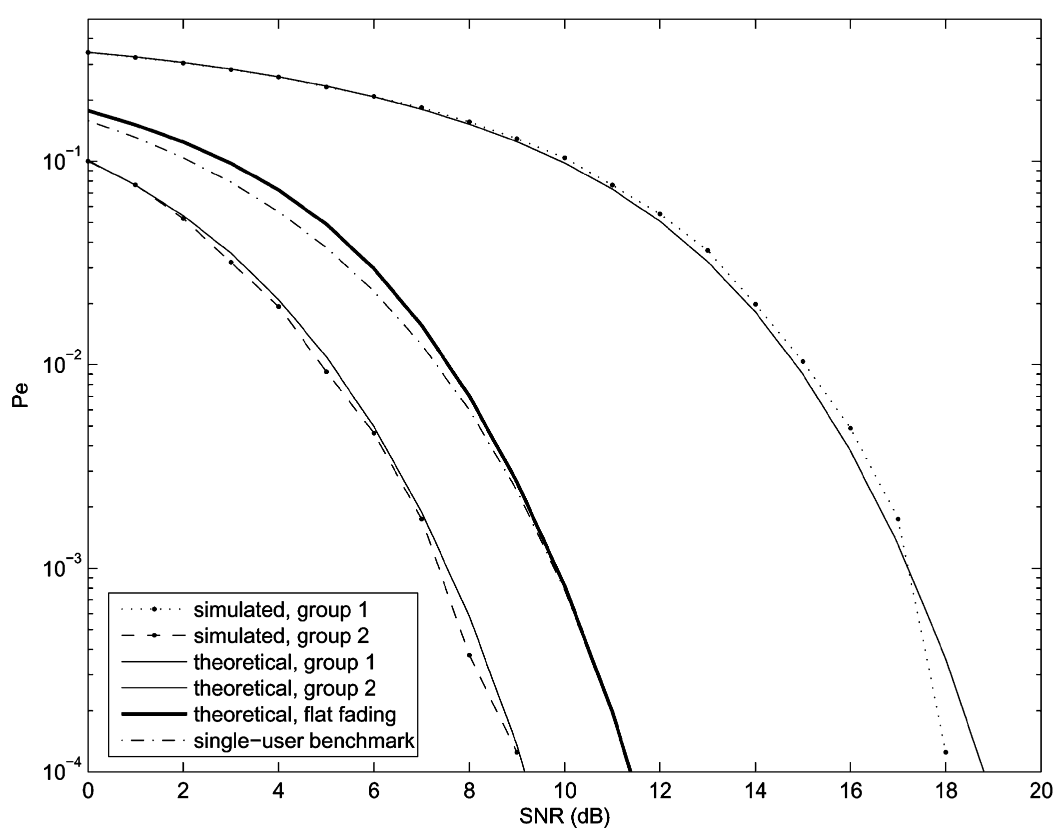

Fig. 7. MC-CDMA with optimal detection. Simulated and theoretical bit-error rates, $K = 16, L = 64$.

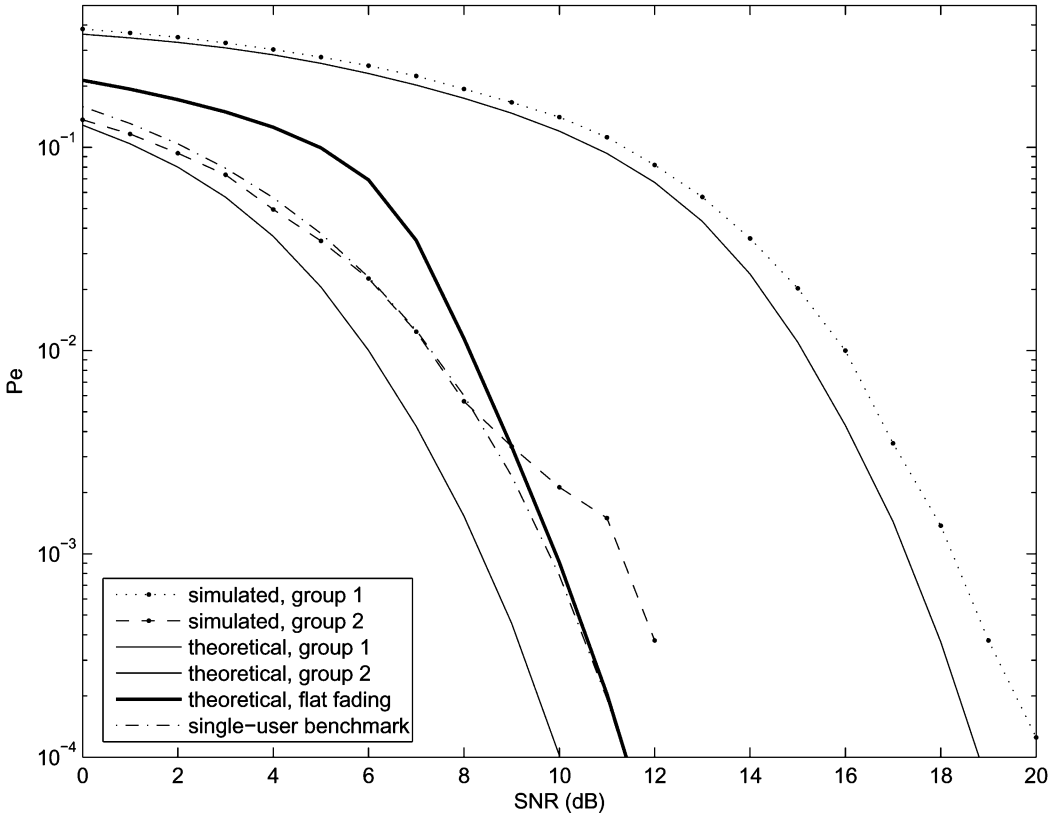

Fig. 8. MC-CDMA with optimal detection. Simulated and theoretical bit-error rates, $K = 16$, $L = 16$.

input and linear MMSE detection is simulated. In the uplink, homogeneous fading is assumed. Hence, the uplink performance is essentially the same as flat fading. The multiuser efficiency is obtained using Proposition 1 and the SER is obtained as that of a single-user Gaussian channel with the same input and an SNR degradation equal to the multiuser efficiency. The simulated curve is found to be very close to the theoretical results.

Fig. 4 shows the uplink and downlink performance of a 16-user system with spreading factor 64. Binary phase-shift keying (BPSK) input is assumed. The rest of the settings is the same as that in Fig. 3 except that individually optimal detection is considered. Both the simulated and theoretical error probabilities are found to be very close to single-user performance in this case.

The remaining experiments assume that the users can be divided into two groups of equal population. The SNRs of all users in each group are the same, while the SNR of users in group 1 is 10 dB higher than that of the users in group 2. For simplicity, we consider downlink communication, where the fading coefficients are the same for all users, although independent over the subcarriers.

Figs. 5 and 6 show the error performance of linear MMSE detection. QPSK input is assumed and the load is assumed to be 1. Fig. 5 plots the performance of an eight-user system while Fig. 6 plots the performance of a 64-user system. For comparison, we also include the theoretical result assuming equal-power users as well as the single-user benchmark. The theoretical results fit well with the simulated SERs for low to moderate SNRs. It is clear that the accuracy of theoretical prediction improves as the number of users increases.

Figs. 7 and 8 show the error performance of optimal posterior mean estimation, both the simulated and predicted by the Propositions 1 and 3. BPSK input is assumed. Fig. 7 plots the performance of a 16-user system with 64 subcarriers. The theoretical prediction is seen to be very good in a wide range of SNRs. Fig. 8 plots the performance for 16 users with 16 subcarriers. A comparison of the two figures indicates that the theoretical prediction tends to be moare accurate for smaller loads.

VI. Conclusion

This correspondence reports new results for the error performance and information rate of MC-CDMA with generic multiuser detection under frequency-selective fading. Analytical results are obtained using statistical physics techniques following an earlier work on DS-CDMA in [18]. Under arbitrary inputs and SNRs, the subchannel for each user formed by the MC-CDMA channel and the detection front end is found to be equivalent to a Gaussian single-user channel. The multiuser efficiencies of all users satisfy a set of joint equations. The new "decoupling principle" considerably simplifies the performance evaluation of MC-CDMA.

Appendix
A Sketch of the Replica Analysis

The reader is referred to [18] for a description of the replica method and a detailed treatment of DS-CDMA using replicas. The replica analysis for MC-CDMA follows the major steps as prescribed in [18], with additional consideration of frequency-selective fading. This appendix presents a sketch of the proof of Propositions 1–3 which highlights divergence from the DS-CDMA problem.

The capacity of the vector Gaussian fading channel (1) under optimal joint decoding can be expressed in the free energy of the system:

$$\begin{aligned}&\frac{1}{L}I(\boldsymbol{X};\boldsymbol{Y}\mid\boldsymbol{H}\circ\boldsymbol{S})\\&\quad=-\frac{1}{L}\mathsf{E}\left\{\log p_{\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{Y}\mid\boldsymbol{H}\circ\boldsymbol{S})\mid\boldsymbol{H}\circ\boldsymbol{S}\right\}-\log(2\pi e)\\&\quad\to\mathcal{F}-\log(2\pi e)\end{aligned}$$

in probability, where the free energy (per chip) can be obtained as

$$\mathcal{F} = -\lim_{K\to\infty}\frac{\beta}{K}\lim_{u\to 0}\frac{\partial}{\partial u}\log \mathsf{E}\left\{p^u_{\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S})\right\}. \tag{12}$$

The key to the replica analysis is to find the moment in (12) for arbitrary positive integer u and then continue the resulting expression to the vicinity of $u=0$ to obtain the derivative.

For an arbitrary positive integer u, the expectation in (12) can be rewritten as an expectation over replicas of the input symbols $\underline{\boldsymbol{X}} = \{X_{ak}|a=0,\dots,u,\ k=1,\dots,K\}$

$$\mathsf{E}\left\{p^u_{\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S})\right\} = \int \mathsf{E}\left\{\prod_{a=0}^{u} p_{Y|\boldsymbol{X},\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{y}\mid \boldsymbol{X}_a, \boldsymbol{H}\circ\boldsymbol{S})\right\}\mathrm{d}\boldsymbol{y} \tag{13}$$

where $X_{0k}\sim p_X$ and $X_{ak}\sim q_X$ for $a=1,\dots,u$ and $k=1,\dots,K$ are all mutually independent. Note that the probability density functions in (13) are all Gaussian. Also, the expectation over the subcarriers within each group are symmetric (equal).

It is assumed that the L subcarriers form M equally populated groups, where all subcarriers from each group are subject to the same fading coefficient. We also assume that the fading process varies slowly over time and can be tracked accurately at the receiver.

A. Downlink

Consider first the downlink, where the fading process is the same for all users. We study the performance conditioned on that the fading coefficients $|H_{lk}|^2 = w_m$ for all subcarriers l in group m, i.e., $l\in G_m$.

For every $a=0,\dots,u$, $m=1,\dots,M$, let

$$V_{ma} = \frac{1}{\sqrt{K}}\sum_{k=1}^{K}\sqrt{\mathsf{snr}_k}(H_{lk}S_{lk})X_{ak} \tag{14}$$

for some $l\in G_m$. The expectation in (13) can then be expressed as an expectation over random variables V_{ma}, which converge as $K\to\infty$ weakly to an array of jointly Gaussian random variables conditioned on the fading coefficients, the SNRs, and the symbols due to the central limit theorem. Let their correlations be denoted by

$$\begin{aligned}\boldsymbol{Q}^{(m)}_{ab} &= \mathsf{E}\left\{V_{ma}V^*_{mb}\mid \boldsymbol{A}, \underline{\boldsymbol{X}}\right\}\\ &= \frac{1}{K}\sum_{k=1}^{K}\mathsf{snr}_k w_m X_{ak}X^*_{bk}\end{aligned} \tag{15}$$

for $a,b=1,\dots,u$, $m=1,\dots,M$. It is convenient to regard $\{\boldsymbol{Q}^{(m)}\}$ as a set of $(u+1)\times(u+1)$ random matrices. Taking the average over the spreading matrix $\boldsymbol{S}$ and then integrate over the received signal $\boldsymbol{y}$, conditioned on the fading coefficients and the correlation matrices, one obtains that

$$\begin{aligned}&\frac{1}{K}\log\mathsf{E}\left\{p^u_{\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S})\right\}\\ &\quad= \frac{1}{K}\log\mathsf{E}\left\{\prod_{m=1}^{M}\exp\left[G^{(u)}(\boldsymbol{Q}^{(m)})\right]\right\} + \mathcal{O}\left(\frac{1}{K}\right)\end{aligned} \tag{16}$$

where

$$G^{(u)}(\boldsymbol{Q}) = -\log\det(I+\boldsymbol{\Sigma}\boldsymbol{Q}) - \log\left(1+\frac{u}{\sigma^2}\right) - u\log(2\pi\sigma^2).$$

Here

$$\boldsymbol{\Sigma} = \frac{\beta}{\sigma^2+u}\left[\begin{array}{c:c} u & -\boldsymbol{e}^\top \\ \hdashline -\boldsymbol{e} & \left(1+\frac{u}{\sigma^2}\right)\boldsymbol{I} - \frac{1}{\sigma^2}\boldsymbol{e}\boldsymbol{e}^\top\end{array}\right]$$

where $\boldsymbol{e}$ is a $u\times 1$ column vector whose entries are all 1. It is important to note that the $(u+1)$-dimentional square matrix $\boldsymbol{\Sigma}$ is symmetric over all replicated dimensions, namely, $\Sigma_{a1}=\Sigma_{a'1}$, $\Sigma_{1a}=\Sigma_{1a'}$, $\Sigma_{aa}=\Sigma_{a'a'}$, $\Sigma_{aa'}=\Sigma_{bb'}$, for all $a,a',b,b'=2,\dots,u+1$ with $a\neq a'$ and $b\neq b'$. In order to evaluate the expectation in (16), we study the joint probability measure of the matrices $\boldsymbol{Q}^{(m)}$. Using Cramér's theorem and Varadhan's theorem [28], one can write

$$\begin{aligned}&\lim_{K\to\infty}\frac{\beta}{K}\log\mathsf{E}\left\{p^u_{\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{H}\circ\boldsymbol{S})\right\}\\ &\qquad= \sup_{\{\boldsymbol{Q}^{(m)}\}}\left\{\frac{1}{M}\sum_{m=1}^{M}G^{(u)}(\boldsymbol{Q}^{(m)}) - \beta I^{(u)}(\{\boldsymbol{Q}^{(m)}\})\right\}\end{aligned}$$

where $I^{(u)}(\{\boldsymbol{Q}^{(m)}\})$ is the rate of the probability measure of the correlation matrices $\{\boldsymbol{Q}^{(m)}\}$ obtained using the Legendre–Fenchel transform

$$\begin{aligned}I^{(u)}(\{\boldsymbol{Q}^{(m)}\}) = \sup_{\{\tilde{\boldsymbol{Q}}^{(m)}\}}&\left\{\sum_{m=1}^{M}\mathsf{tr}\left\{\boldsymbol{Q}^{(m)}\tilde{\boldsymbol{Q}}^{(m)}\right\}\right.\\ &\left.-\log\mathsf{E}\left\{\exp\left[\mathsf{snr}\boldsymbol{X}^\top\left(\sum_{m=1}^{M}w_m\tilde{\boldsymbol{Q}}^{(m)}\right)\boldsymbol{X}\right]\right\}\right\}.\end{aligned}$$

Seeking the supremum by setting the derivatives with respect to $\boldsymbol{Q}^{(m)}$ and $\tilde{\boldsymbol{Q}}^{(m)}$ to 0, one finds that the correlation matrices satisfy the following set of joint equations:

$$\begin{aligned}\boldsymbol{Q}^{(m)} &= \frac{\mathsf{E}\left\{\mathsf{snr}\,w_m\boldsymbol{X}\boldsymbol{X}^\top\exp\left[\mathsf{snr}\boldsymbol{X}^\top\left(\sum_{m=1}^{M}w_m\tilde{\boldsymbol{Q}}^{(m)}\right)\boldsymbol{X}\right]\right\}}{\mathsf{E}\left\{\exp\left[\mathsf{snr}\boldsymbol{X}^\top\left(\sum_{m=1}^{M}w_m\tilde{\boldsymbol{Q}}^{(m)}\right)\boldsymbol{X}\right]\right\}},\\ \tilde{\boldsymbol{Q}}^{(m)} &= -\frac{1}{\beta M}\left(\boldsymbol{I}+\boldsymbol{\Sigma}\boldsymbol{Q}^{(m)}\right)^{-1}\boldsymbol{\Sigma}.\end{aligned}$$

Let $\tilde{\boldsymbol{Q}}=\sum_{m=1}^{M}w_m\tilde{\boldsymbol{Q}}^{(m)}$ and $\boldsymbol{Q}=\boldsymbol{Q}^{(m)}/w_m$, the above equations can be simplified to

$$\boldsymbol{Q} = \frac{\mathsf{E}\left\{\mathsf{snr}\boldsymbol{X}\boldsymbol{X}^\top\exp\left[\mathsf{snr}\boldsymbol{X}^\top\tilde{\boldsymbol{Q}}\boldsymbol{X}\right]\right\}}{\mathsf{E}\left\{\exp\left[\mathsf{snr}\boldsymbol{X}^\top\tilde{\boldsymbol{Q}}\boldsymbol{X}\right]\right\}} \tag{17a}$$

$$\tilde{\boldsymbol{Q}} = -\frac{2}{\beta M}\sum_{m=1}^{M}\left(\frac{1}{w_m}\boldsymbol{I}+\boldsymbol{\Sigma}\boldsymbol{Q}\right)^{-1}\boldsymbol{\Sigma}. \tag{17b}$$

The complete solution set to the joint (17) is not known. However, it is possible to obtain solutions that satisfy replica symmetry. Consider individually optimal detection for simplicity, where $\boldsymbol{Q}$ and $\tilde{\boldsymbol{Q}}$ are symmetric over all replica indices. Precisely

$$\boldsymbol{Q} = \begin{bmatrix} p & & m \\ & \ddots & \\ m & & p\end{bmatrix}, \quad \tilde{\boldsymbol{Q}} = \begin{bmatrix} c & & d \\ & \ddots & \\ d & & c\end{bmatrix}$$

where m, p, c, d are some real numbers. The reason for choosing the replica symmetric solution is discussed in [18]. Solving the four parameters from (17), we find that $c = 0$, and

$$d = \frac{1}{M} \sum_{m=1}^{M} w_m (1 + \beta w_m (p - m))^{-1}$$

where m and p have physical meanings associated with the single-user channel as defined in Fig. 1(b) and its retrochannel. In particular, m is obtained as the average correlation of the input and the retrochannel output of all users and p the average second moment. Moreover, by defining $\eta = d$, we find that

$$p - m = \mathsf{E}\left\{\mathsf{snr}\ \mathsf{mmse}(\eta\, \mathsf{snr})\right\}.$$

Finally, the joint (17) is reduced to the fixed point (4). The downlink part of Proposition 1 is thus established.

For a generic multiuser detector, symmetry is restricted to the replicated indices $a = 2, \ldots, u + 1$. The joint (17) become (11) in Proposition 3 with $\bar{\boldsymbol{W}} = [w_1, \ldots, w_M]$.

B. Uplink

Consider the uplink where the fading process is independent for all users. It is still assumed that for each user the subcarriers are grouped. The analysis is similar to that for the downlink. Let V_{ma} be defined the same as in (14). Note that for each subcarrier l, the fading coefficients for all users are independent. The correlations are then defined as

$$\begin{aligned}\boldsymbol{Q}_{ab}^{(m)} &= \mathsf{E}\left\{V_{ma} V_{mb}^{*} \mid \boldsymbol{W}, \boldsymbol{A}, \underline{\boldsymbol{X}}\right\} \\ &= \frac{1}{K} \sum_{k=1}^{K} \mathsf{snr}_k W_{mk} X_{ak} X_{bk}\end{aligned}$$

where the difference from (15) is that W_{mk} is a random variable which varies with the user number. The rest of the analysis is essentially the same as that in Section A of the Appendix, only with w_m replaced by a random variable W_{mk}. Let W_m represent a random variable with the mixture distribution of W_{mk}, $k = 1, \ldots, K$. In parallel to (17), a set of fixed-point equations is obtained

$$\boldsymbol{Q}^{(m)} = \frac{\mathsf{E}\left\{W_m \mathsf{snr} \boldsymbol{X}\boldsymbol{X}^{\top} \exp\left[\mathsf{snr} \boldsymbol{X}^{\top} \sum_{m=1}^{M} W_m \tilde{\boldsymbol{Q}}^{(m)} \boldsymbol{X}\right]\right\}}{\mathsf{E}\left\{\exp\left[\mathsf{snr} \boldsymbol{X}^{\top} \sum_{m=1}^{M} W_m \tilde{\boldsymbol{Q}}^{(m)} \boldsymbol{X}\right]\right\}}$$

$$\tilde{\boldsymbol{Q}}^{(m)} = -\frac{2}{\beta M} \sum_{m=1}^{M} \left(\boldsymbol{I} + \boldsymbol{\Sigma} \boldsymbol{Q}^{(m)}\right)^{-1} \boldsymbol{\Sigma}.$$

Assuming replica symmetry, the above fixed-point equations become (11) in Proposition 3 by defining an appropriate efficiency function $\eta(\cdot)$ and an auxiliary function $\xi(\cdot)$. In the special case of individually optimal detection, the result reduces to (5) in Proposition 1.

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