Problem 1:

We use Erlang formula B to compute the blocking probability – we compute for various values of m until we reach the value of $P_b = 0.01$ or lower:

$$P_b = \frac{\frac{\rho^m}{m!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}$$, where ρ = 0.2n and m is the number of lines.

Here we use Excel, and start a table with various values of m and compute blocking probability until we get the right number that is the closest to 0.01. For n=10 we start with m=10, 9, 8, 7. For n=100, we compute for m in the range of 20 to 40, and so on for higher values of n. For n=1,000 and 4,000 we use the approximation given in the notes. It even works reasonably well for n=100. (I found that the approximation formula works even for n=100).

For n=10, ρ =2, we need m = 7 lines

For n=100 ρ =20, we need m = 30 lines

For n=1000 ρ =200, we need m = 221

For n=4000 ρ =800, we need m = 829 (only 29 more than the average).

Problem 2:

X and Y are random variables taking values and probabilities given by a formula, showing 5 values for X and 4 values for Y. Hence, we need a 4 by 6 table and we use Excel to tabulate the data and solve the problem.

a. Find the means, variances, and correlation for X and Y.

We first fill in the joint probabilities (from the formula) and find the marginal probabilities (by adding rows and columns) in the table to obtain:

	X=0	X=1	X=2	X=3	X=4	P(Y=m)
Y=0	0.2	0.027067	0.003663	0.000496	0	0.231226
Y=2	0.027067	0.2	0.027067	0.003663	0	0.257797
Y=4	0.003663	0.027067	0.2	0.027067	0	0.257797
Y=6	0.000496	0.003663	0.027067	0.2	0.022	0.253226
P(X=k)	0.231226	0.257797	0.257797	0.231226	0.022	1.000046

$$\mu_X = \sum_{k=0}^{4} k\, P\{X=k\} = 1.55507$$

$$\mu_Y = \sum_{k=0}^{3} (2k)\, P\{Y=2k\} = 3.066139$$

$$Var(X) = \sum_{k=0}^{4} (k-\mu_X)^2\, P\{X=k\} = 1.303778,\ \sigma_X = 1.141831$$

$$Var(Y) = \sum_{k=0}^{3} (2k-\mu_Y)^2\, P\{Y=2k\} = 4.87087,\ \sigma_Y = 2.207005$$

Correlation between X and Y:

$$r_{XY}\,\sigma_X\,\sigma_Y = \sum_{k=0}^{4}\sum_{m=0}^{3} (k-\mu_X)(2m-\mu_Y)P\{X=k \text{ and } Y=2m\} =$$

$$\left[\sum_{k=0}^{4}\sum_{m=0}^{3} (k)(2m)P\{X=k \text{ and } Y=2m\}\right] - \mu_X\mu_Y = 2.270044$$

The double summation above is shown in the following table copied from Excel, where each cell is the product of the values of X and Y and their probabilities as shown in the previous table. The number in the last corner is equal to R_{XY}.

	X=0	X=1	X=2	X=3	X=4	Add columns
Y=0	0	0	0	0	0	0
Y=2	0	0.4	0.108268	0.021979	0	0.530247
Y=4	0	0.108268	1.6	0.324805	0	2.033073
Y=6	0	0.021979	0.324805	3.6	0.528	4.474783
Add rows	0	0.530247	2.033073	3.946783	0.528	7.038103

$C_{XY} = r_{XY}\,\sigma_X\,\sigma_Y = 7.038103 - \mu_X\mu_Y = 2.270044$

$r_{XY} = 0.900802$ (a very strong correlation)

$r^2 = 0.811443$

b. Find regressions for estimating X from Y and Y from X.

$$\hat{Y} = \mu_Y + (r_{XY}\sigma_Y/\sigma_X)(X-\mu_X) = 0.358565 + 1.741127X$$

$$\hat{X} = \mu_X + (r_{XY}\sigma_X/\sigma_Y)(Y-\mu_Y) = 0.126111 + 0.466045Y$$

c. The MSE for estimating Y from X is = $Var(Y)(1-r^2)$ = 4.87087 (1–0.811443) = 0.918435

The MSE for estimating X from Y is = $Var(X)(1-r^2)$ = 1.303778 (1–0.811443) = 0.245836

Problem 3 (a)

We complete the table to find the marginal probabilities by adding rows and columns:

P(X=m,Y=n)	Y=–2	Y=–1	Y=1	Y=2	P(X=m)
X=–1	1/8	0	0	0	1/8
X=–0.5	0	1/4	0	0	1/4
X=0.5	0	0	1/2	0	1/2
X=1	0	0	0	1/8	1/8
P(Y=n)	1/8	1/4	1/2	1/8	

Means:
$\mu_X = -1\times (1/8)+ -0.5\times (1/4)+0.5\times (1/2)+1\times (1/8)=0.125$
$\mu_Y = -2\times (1/8) -1\times (1/4)+1\times (1/2)+2\times (1/8)=0.25$

Variances:
$Var(X) = 1\times (1/8)+ 0.25\times (1/4) +0.25\times (1/2)+1\times (1/8)-(1/8)^2 = 0.4219$, $\sigma_X = 0.6495$
$Var(Y) = 4\times (1/8)+ 1\times (1/4)+ 1\times (1/2) +4\times (1/8) - (1/4)^2 =1.6875$, $\sigma_Y = 1.299$

Correlation coefficient:
$r_{XY}\sigma_X\sigma_Y = -1\times (-2) \times (1/8)+(-0.5) \times (-1) \times (1/4)+0.5\times1\times (1/2)+1\times2\times (1/8) - (0.125)(0.25) = 0.875-0.03125=0.8438$

$r_{XY} = 0.8438/(0.6495\times1.299)=1.000$

Perfectly correlated (all values fall on a straight line!)

Problem 3(b):

Marginal probabilities obtained by adding columns and rows:

P(X=m,Y=n)	Y=–2	Y=–1	Y=0	Y=1	Y=2	P(X=m)
X=0	0	0	0.25	0	0	0.25
X=1	0	0.125	0	0.125	0	0.25
X=4	0.25	0	0	0	0.25	0.5
P(Y=n)	0.25	0.125	0.25	0.125	0.25	1.00

Means:
$\mu_X = 0\times (1/4)+ 1\times (1/4)+4\times (1/2) =2.25$
$\mu_Y = -2\times (1/4) -1\times (1/8) +0\times (1/4)+1\times (1/8) +2\times (1/4)=0$

Variances:

$Var(X) = 0\times(1/4)+1\times(1/4)+16\times(1/2) -(2.25)^2 = 051/16$,
$Var(Y) = 4\times(1/4)+1\times(1/8)+0\times(1/4)+1\times(1/8)+4\times(1/4) = 2.25$,

Covariance and correlation coefficient:

$r_{XY}\sigma_X\sigma_Y = -2(4)(1/4) +(-1)(1)(1/8)+(0)(0)(1/4)+(1)(1)(1/8)+(2)(4)(1/4) = 0$

$r_{XY} = 0$

The correlation is zero, but we see that $X=Y^2$, so if we know the value of Y we can find X exactly, while if we know the value of X we can only guess at the value of Y with 50% of being right as it can take positive or negative values.

4. Problem solution is shown in an excel file, but here are the answers:

	X	Y	Z	W
Mean	1.036	1.56125	1.56775	1.56675
Std Dev	0.654972	0.626677	0.529685	0.518951
Correlation Coefficient With X		0.913968	0.693427	0.472621
Slope	a	0.955235	0.857444	0.596499
Intercept	b	-0.45536	-0.30826	0.101434

	Y	Z	W
Residual Variance	0.26699	0.470511	0.573181
Residual Standard Deviation	0.265779	0.471925	0.577205

In the excel workbook posted on the web ("correlation4.xls") the scatter diagrams and the regression lines are shown on sheet #2. Sheet #1 has all the computations.