Benefits of Limited Feedback for Wireless Channels*

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Abstract

The benefits of limited feedback are studied in the context of some different wireless channel models. Namely, we consider using the feedback for signature optimization in Code-Division Multiple Access (CDMA) and multi-antenna channels, and power and rate optimization for multi-carrier transmission. Our results are asymptotic as system dimensions tend to infinity, and explicitly characterize the performance (e.g., output Signal-to-Interference-Plus Noise Ratio (SINR) or capacity) as a function of feedback bits per degree of freedom (e.g., processing gain or product of number of receive antennas times transmit antennas). Numerical examples show that the asymptotic analysis accurately predicts the performance of finite-size systems of interest.

1 Introduction

The performance, or achievable rate, associated with a wireless link generally depends on how much the transmitter knows about the channel and interference at the receiver. Namely, this information can influence the allocation of available resources, such as power and rate, across available degrees of freedom to exploit channel conditions and avoid interference.

In this paper, we give an overview of some recent work characterizing the benefits of limited feedback in the context of a few wireless channel models.[1] We emphasize that *limited* feedback is to be distinguished from *partial* feedback. Namely, limited feedback means that the feedback channel has a finite data rate, so that in a given time interval, the receiver relays some finite number of bits back to the transmitter. In contrast, partial feedback typically means that the receiver sends back a measurement, or statistical information about the channel. For example, partial feedback may take the form of a channel estimate [1], or second-order statistics [2, 3]. Here we consider only limited feedback assuming that the channel and interference are stationary or slowly varying, and ignore any channel estimation error.

The channel models we consider pertain to CDMA, space-time MIMO channels, and multi-carrier (MC) signaling. In each case, the received signal is a vector given by the

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[1]See `http://www.ece.northwestern.edu/~mh/feedback` for a list of references.

same general linear model, and the objective is to use the feedback to select signatures (in the case of CDMA and MIMO channels), along with associated powers and rates. Differences among the models are reflected in how the channel and signatures are defined in time, space, and frequency.

For each scenario, our objective is to compute the performance measure (namely, Signal-to-Interference Plus Noise Ratio (SINR) or capacity) as a function of the number of feedback bits per degree of freedom (e.g., processing gain in the case of CDMA, or number of subcarriers for MC signaling). Our results are asymptotic as the degrees of freedom tend to infinity. Comparisons with simulation results show that the asymptotic results presented here can accurately predict the performance of finite-size systems of interest.

In the case of CDMA, we specify a feedback scheme, which we call "Random Vector Quantization (RVQ)", which is optimal, i.e., achieves an upper bound on received asymptotic SINR over all possible feedback schemes. Furthermore, by applying results from the theory of extreme order statistics, this SINR can be explicitly computed for the matched filter receiver, and approximated for the Minimum Mean Squared Error (MMSE) linear receiver. For the other models, we specify achievable rates with some specific feedback schemes. Namely, for MC signaling, the feedback consists of which carriers to activate along with the number of bits per activated carrier.

Related work on the performance of beamforming for space-time MIMO channels with limited feedback has been presented in [4–7]. The emphasis in that work is on finding optimal quantization schemes for spatial signatures for finite-size systems. (References [4, 5] only consider the Multi-Input/Single-Output (MISO) channel.) Additional work in [6, 8] characterizes capacity for MIMO channels with different limited feedback schemes, again for finite-size systems. For the MISO channel, RVQ is again asymptotically optimal, so that our performance results predict the performance of other optimal feedback quantization schemes for systems with enough degrees of freedom.

2 System Model

We start with the basic discrete-time model for a vector channel with $N \times 1$ received vector

$$\boldsymbol{r}(i) = \boldsymbol{H}\boldsymbol{S}\boldsymbol{A}\boldsymbol{b}(i) + \boldsymbol{n}(i) \tag{1}$$

at time i, where $\boldsymbol{b}(i) = [b_1(i), \cdots, b_K(i)]^T$ is a $K \times 1$ vector of transmitted symbols, $\boldsymbol{A}$ is a $K \times K$ diagonal amplitude matrix, $\boldsymbol{S}$ is an $N \times K$ signature matrix, $\boldsymbol{H}$ is an $N \times N$ channel matrix, and the noise $\boldsymbol{n}(i)$ is Gaussian with covariance matrix $\sigma^2\boldsymbol{I}$. At the receiver, we will assume that the received vector is filtered by the $N \times K$ matrix $\boldsymbol{C}$,

$$\boldsymbol{y}(i) = \boldsymbol{C}^\dagger\boldsymbol{r}(i) \tag{2}$$

where $\dagger$ indicates Hermitian transpose. Our objective is to optimize a performance measure, i.e., either received SINR or capacity, given that the receiver can feedback B_N bits to the transmitter.

We will consider the following systems, corresponding to different choices for $\boldsymbol{S}$, $\boldsymbol{H}$, and $\boldsymbol{C}$. In all cases, the receiver is assumed to have perfect channel knowledge, and the transmitted signal is constrained in average power.

CDMA: The symbols are transmitted from K users, who cannot coordinate their choice of code words, although we assume synchronous transmissions. The k^{th} column of $\boldsymbol{S}$ is $\boldsymbol{s}_k$, the spreading sequence for user k, and the channel and amplitude matrices $\boldsymbol{H} = \boldsymbol{A} = \boldsymbol{I}_N$, the $N \times N$ identity matrix. Our objective is to select the signature for user 1, $\boldsymbol{s}_1$, to maximize the received SINR in the presence of other random signatures with *i.i.d.* elements.

Single-User Multiple-Input-Multiple-Output (MIMO) Channel: The K symbols emanate from a single source, and the transmitted symbol streams across transmit antennas correspond to the same code word. The elements of the channel matrix are assumed to be *i.i.d.* complex Gaussian, corresponding to K transmit antennas and N receiver antennas with flat fading Rayleigh channels between transmitter-receiver antenna pairs. The objective is to select the signature and amplitude matrices to maximize the associated capacity. We will assume an optimal receiver, so that $\boldsymbol{C} = \boldsymbol{I}$.

Single-User Multi-Carrier (MC) Transmission: The signature matrix $\boldsymbol{S}$ and receiver matrix $\boldsymbol{C}$ are the inverse DFT and channel matrices, respectively, and the channel $\boldsymbol{H}$ is circulant. Hence the overall channel matrix $\boldsymbol{C}^\dagger \boldsymbol{H}\boldsymbol{S}$ is diagonal, where the k^{th} diagonal element is the channel gain for the k^{th} carrier, or sub-channel. The elements of $\boldsymbol{b}(i)$ are the symbols transmitted through the different sub-channels. The objective is then to select the amplitudes $\boldsymbol{A}$ and bits per sub-channel to maximize the associated capacity. Since the receiver does not have to specify signatures, the feedback needed to achieve the optimal growth in capacity vs. N is substantially less than for previous cases.

3 CDMA

In this case the model reduces to $\boldsymbol{r} = \boldsymbol{S}\boldsymbol{b} + \boldsymbol{n}$, and our objective is to specify the signature $\boldsymbol{s}_1$ that maximizes the received SINR. With a matched filter (MF), $\boldsymbol{c}_1 = \boldsymbol{s}_1$ (first column of $\boldsymbol{C}$), and the SINR for user 1 is $\gamma_1 = 1/(\sigma^2 + \sum_{k\neq 1} |\boldsymbol{s}_1^\dagger \boldsymbol{s}_k|^2)$ where we assume $\|\boldsymbol{s}_k\|^2 = 1$ for $k = 1, \cdots, K$. With infinite feedback, we would select $\boldsymbol{s}_1$ to minimize the total interference power $\sum_{k\neq 1} |\boldsymbol{s}_1^\dagger \boldsymbol{s}_k|^2$. In what follows we assume that the interfering signatures $\boldsymbol{s}_k$, $k \neq 2$, are random *i.i.d.*.

Given B_N feedback bits, we can choose one of 2^{B_N} signatures from the codebook, or quantization set $\mathcal{V} = \{\boldsymbol{v}_j; 1 \leq j \leq 2^{B_N}\}$. That is, the receiver selects

$$\hat{\boldsymbol{s}}_1 = \arg\max_{\boldsymbol{v}\in\mathcal{V}} \gamma_1(\boldsymbol{v})$$

where $\gamma_1(\boldsymbol{v})$ is the SINR for user 1 with $\boldsymbol{s}_1 = \boldsymbol{v}_1$. The problem is how to design the codebook $\mathcal{V}$, which maximizes the SINR. This optimization depends on the distribution of the interfering signatures, and is difficult for finite N and K. However, for large K and N, the eigenvectors of the interference plus noise covariance matrix $\boldsymbol{R}_1 = \sum_{k\neq 1} \boldsymbol{s}_k \boldsymbol{s}_k^\dagger + \sigma^2 \boldsymbol{I}$ are isotropically distributed. We will therefore consider choosing the vectors in $\mathcal{V}$ to have *i.i.d.* elements with variance $1/N$, so that $\|\boldsymbol{v}_j\| \to 1$ in the mean square sense as $N \to \infty$. We refer to this scheme as *Random Vector Quantization (RVQ)*.

In general, we can choose the elements of vectors in $\mathcal{V}$ according to any joint distribution $F_{\mathcal{V}}$. With this in mind, let $X_j = \sum_{k\neq 1} (\boldsymbol{v}_j^\dagger \boldsymbol{s}_k)^2$, which has cumulative distribution function (cdf) F_X, and let $W = \min\{X_1, \cdots, X_{2^{B_N}}\}$ be the minimum interference over the set of available (quantized) signatures. The corresponding SINR for user 1 is $\bar{\gamma} = 1/(\sigma^2 + E[W])$. The following theorem states that RVQ is asymptotically optimal, i.e., asymptotically achieves the maximum SINR, where asymptotic refers to the large system limit $(N, K, B_N) \to \infty$ with fixed normalized load $\bar{K} = K/N$ and normalized feedback bits per dimension $\bar{B}_N = B_N/N$. In what follows, the superscript ∞ denotes this large system limit. In particular, $\gamma_{\text{rvq}}^\infty$ is the large system SINR with RVQ.

Theorem 1 *Let $\gamma_{\mathcal{V}}^{\infty}$ denote the large system SINR, where the entire set of signatures $\mathcal{V}$ is chosen from an arbitrary distribution $F_{\mathcal{V}}$. Then*

$$\gamma_{\mathcal{V}}^{\infty} \leq \gamma_{\mathrm{rvq}}^{\infty} = \frac{1}{\sigma^2 + W^{\infty}}$$

where

$$W^{\infty} = \lim_{(K,N,B_N)\to\infty} E[W] = \lim_{(K,N,B_N)\to\infty} F_X^{-1}\left(\frac{1}{2^{B_N}}\right), \tag{3}$$

and the latter limit converges for $0 \leq \bar{K} \leq 1$ and $\bar{B}_N \geq 0$.

The proof relies on the asymptotic theory of extreme statistics [9]. The cdf F_X is a Gamma distribution with parameters N and K, and (3) can be numerically evaluated to determine $\gamma_{\mathrm{rvq}}^{\infty}$. An example showing the asymptotic SINR for RVQ vs. feedback bits per dimension B_N/N is shown in Figure 1 for $\bar{K} = 3/4$ and background SNR=8 dB. (See curves corresponding to MF.) Note that with 1 feedback bit per dimension, the SINR is within 1 dB of the single-user performance, and is substantially higher than the SINR obtained by feeding back a scalar quantized version of the optimal signature.

With a linear Minimum Mean Squared Error (MMSE) receiver, the SINR for user 1 with RVQ is given by $\beta_{\mathrm{rvq}} = \max\{\beta_1, \cdots, \beta_{2^{B_N}}\}$ where $\beta_j = \boldsymbol{v}_j^{\dagger}\boldsymbol{R}_1^{-1}\boldsymbol{v}_j$, which has cdf F_{β}. The asymptotic SINR is again maximized with RVQ, and is given by the following Theorem.

Theorem 2 *Let $\beta_{\mathcal{V}}^{\infty}$ denote the large system SINR with an MMSE receiver, where the quantized signature set $\mathcal{V}$ is chosen from distribution $F_{\mathcal{V}}$. Then*

$$\beta_{\mathcal{V}}^{\infty} \leq \beta_{\mathrm{rvq}}^{\infty} = \lim_{(K,N,B_N)\to\infty} F_{\beta}^{-1}\left(1 - \frac{1}{2^{B_N}}\right)$$

for $0 \leq \bar{K} \leq 1$ and $\bar{B}_N \geq 0$.

The cdf F_{β} for the SINR at the output of MMSE receiver with random signatures is not known for finite K and N, but is asymptotically Gaussian [10]. Expressions for the asymptotic mean β^* and normalized variance σ_{β}^2/N are also given in [10]. Approximating F_{β} with this Gaussian distribution and applying Theorem 2 gives the following approximation for $\beta_{\mathrm{rvq}}^{\infty}$,

$$\tilde{\beta}_{\mathrm{rvq}}^{\infty} = \beta^* + \sigma_{\beta}\sqrt{(2\log 2)\bar{B}_N}. \tag{4}$$

Figure 1 compares the asymptotic SINR vs. $\bar{B}_N$ computed from (4) with simulated values for $N = 16$. Note that $\tilde{\beta}_{\mathrm{rvq}}^{\infty} \to \infty$ with $\bar{B}_N$, whereas $\beta_{\mathrm{rvq}}^{\infty}$ is upper bounded by the single-user SNR. This is because F_{β}, which has compact support, is being approximated with a Gaussian distribution, which does not have compact support.

Because the number of vectors in $\mathcal{V}$ increases exponentially with B_N, RVQ is impractical even for relatively small B_N. A simpler reduced-rank signature optimization scheme, in which the signature is constrained to lie in a lower-dimensional subspace, is analyzed in [11]. With scalar quantization (SQ) of the reduced-rank coefficients, this scheme can perform close (e.g., within 1 dB) to the RVQ bound.

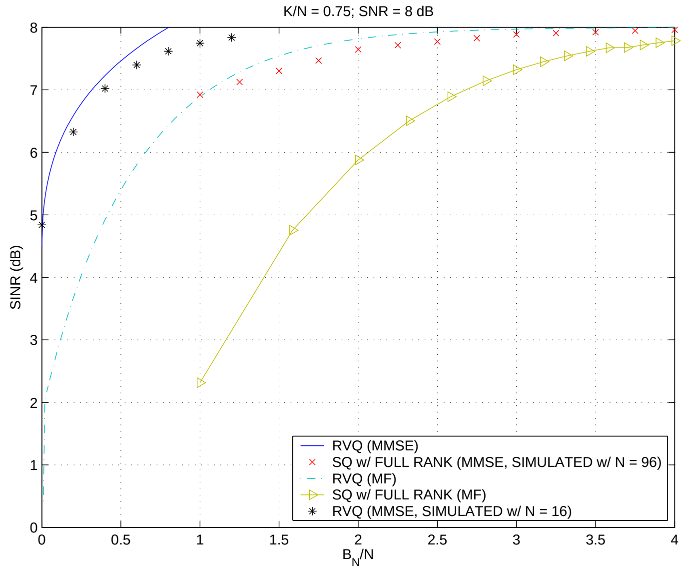

Figure 1: SINR in dB versus feedback bits per dimension with MF and MMSE receivers.

4 Single-User MIMO Channel

We now consider a single-user channel with K transmit antennas and N receive antennas. Letting $\boldsymbol{x} = \boldsymbol{SAb}$, and assuming that the channel is known at the receiver, the normalized channel capacity is $C_N = \frac{1}{N} \log\det\left(\boldsymbol{I}_N + \frac{1}{\sigma^2}\boldsymbol{HQH}^\dagger\right)$ where $\boldsymbol{Q} = E[\boldsymbol{xx}^\dagger] = \boldsymbol{S}|\boldsymbol{A}|^2\boldsymbol{S}^\dagger$ is the covariance matrix for the transmitted signal. We wish to use the feedback to optimize $\boldsymbol{Q}$, assuming the channel matrix $\boldsymbol{H}$ has *i.i.d.* elements. Namely, without any feedback the optimal $\boldsymbol{Q} = \frac{1}{K\sigma^2}\boldsymbol{I}_K$ [12], i.e., power and information bits are allocated equally across transmit antennas. With unlimited feedback, the optimal $\boldsymbol{Q} = \boldsymbol{UDU}^\dagger$ where $\boldsymbol{U}$ is a unitary matrix whose columns are eigenvectors of $\boldsymbol{H}^\dagger\boldsymbol{H}$, and $\boldsymbol{D}$ is a diagonal matrix whose elements are computed by water-filling over the available dimensions.

The B_N feedback bits are used to specify a covariance matrix $\hat{\boldsymbol{Q}}$. Specifically, we can construct a codebook $\mathcal{V} = \{\boldsymbol{V}_1, \cdots, \boldsymbol{V}_{2^{B_N}}\}$, corresponding to RVQ, which contains $K \times D$ independent and isotropic unitary matrices. The covariance matrix is given by $\boldsymbol{Q}_j = \frac{1}{D}\boldsymbol{V}_j\boldsymbol{V}_j^\dagger$ for $1 \le j \le 2^{B_N}$ where D is the rank of $\boldsymbol{Q}_j$, and the receiver determines

$$\hat{\boldsymbol{Q}} = \arg \max_{1 \le j \le 2^{B_N}} \left\{ C_N^{(j)} = \frac{1}{N} \log\det\left(\boldsymbol{I} + \frac{1}{\sigma^2}\boldsymbol{HQ}_j\boldsymbol{H}^\dagger\right)\right\}.$$

We can again evaluate the asymptotic performance in terms of the cdf of C_N^j, which we denote as F_{C_N}.

Theorem 3 *As $(N, K, B_N, D) \to \infty$ with fixed $\bar{N} = N/K > 0$, $\hat{B}_N = B_N/N^2$, and $\bar{D} = D/K$, the sum mutual information per receive antenna $\mathcal{I}_{\rm rvq}^N = \max_j C_N^{(j)}$ converges almost surely to*

$$\mathcal{I}_{\rm rvq}^\infty = \lim_{(N,K,D,B_N)\to\infty} F_{C_N}^{-1}\left(1 - \frac{1}{2^{B_N}}\right). \quad (5)$$

Note that in this case, B_N grows in proportion to N^2, due to the fact that $\hat{\boldsymbol{Q}}$ has $N^2/\bar{N}$ elements. In analogy with the previous CDMA analysis with the MMSE receiver, F_{C_N} is unknown for finite N, but is asymptotically Gaussian [13]. Using the Gaussian cdf gives the approximate asymptotic sum mutual information

$$\tilde{\mathcal{I}}_{\mathrm{rvq}}^{\infty} = \mu + \sigma_\mu \sqrt{(2 \log 2) \hat{B}_N} \tag{6}$$

where μ and σ_μ can be determined from [13, Theorem 1.1]. Note that $\tilde{\mathcal{I}}_{\mathrm{rvq}}^{\infty}$ is a function of both the rank $\bar{D}$ and feedback $\hat{B}_N$, and can be optimized over $\bar{D}$ for a given $\hat{B}_N$. This is illustrated in Figure 2, which shows $\tilde{\mathcal{I}}_{\mathrm{rvq}}^{\infty}$ vs. normalized rank D/K. These curves show that the capacity becomes more sensitive to rank selection as the feedback increases. The right graph shows mutual information vs. $\hat{B}_N$ for different SNRs $1/\sigma^2$. Curves are shown for the optimal rank and full rank $\bar{D} = 1$, and are compared with simulated values with $K = 8$. Again, the asymptotic expression (6), based on the Gaussian approximation for $C_N^{(j)}$, increases beyond the capacity with unlimited feedback. In this case, feedback offers only a modest increase in data rate.

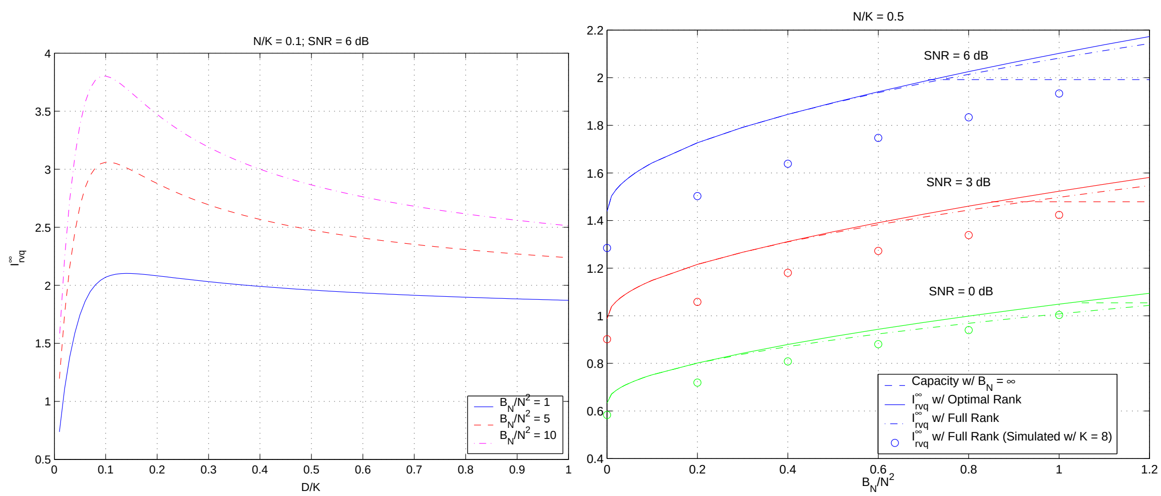

Figure 2: The left graph shows mutual information per receive antenna vs. normalized rank with different amounts of feedback. The right plot shows the mutual information vs. normalized feedback.

5 Multi-Carrier Transmission

In this case the received vector after filtering with $\boldsymbol{C}$ (the FFT matrix) is $\boldsymbol{y} = \tilde{\boldsymbol{H}}\boldsymbol{A}\boldsymbol{b} + \tilde{\boldsymbol{n}}$, where $\tilde{\boldsymbol{H}}$ is diagonal, and for convenience, $E[\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}^\dagger] = \boldsymbol{I}_N$. The diagonal elements of $\tilde{\boldsymbol{H}}$ are the complex random sub-channel coefficients h_i, $i = 1, \cdots, N$. The capacity conditioned on the sub-channel gains is given by $C_N = \sum_{i=1}^{N} \log(1 + P_i \mu_i)$, where $P_i = A_i^2$ is the power assigned to sub-channel i. The feedback can be used to specify the set of powers $\{P_i\}$, i.e., with unlimited feedback, $\{P_i\}$ is determined by water pouring.

In contrast to the previous models, the feedback is not used to specify signatures, which reduces the amount of feedback needed to achieve a capacity close to the capacity with unlimited feedback. An objective, then, is to characterize how the capacity scales with N with different amounts of feedback B_N. We first consider how the capacity scales with unlimited feedback. In what follows, we assume that the sub-channel gain μ_i is a random variable with pdf f_μ and cdf F_μ, and $\bar{F}_\mu(x) = 1 - F_\mu(x)$. Initially, we assume that $\{\mu_i\}$ is $i.i.d.$, and later consider the case where this sequence is correlated.

It will be useful to consider an *on-off* power distribution, in which the transmitter allocates equal power $\bar{P}$ across a subset of sub-channels with gains that exceed a threshold μ_0. The power constraint then becomes $\sum_{i=1}^{N} \bar{P}\ \mathbf{1}_{\mu_i \geq \mu_0} \leq \mathcal{P}$. Optimizing the threshold gives the corresponding on-off capacity for finite N,

$$C_N^{(\text{on-off})} = \max_{\mu_0} \sum_{i=1}^{N} \log\left(1 + \bar{P}\mu_i\right) \mathbf{1}_{\mu_i \geq \mu_0}$$

where $\bar{P}$ depends on μ_0. The optimal threshold that maximizes $C_N^{(\text{on-off})}$ is denoted as μ_0^*.

In what follows, we say that two random sequences $\{x_n\}$ and $\{y_n\}$ are *asymptotically equivalent* if $\lim_{n\to\infty} x_n/y_n = 1$ almost surely, and write $x_n \asymp y_n$. The next theorem specifies the asymptotic scaling for the capacity with water pouring (unlimited feedback), $C_N^{(\text{wf})}$, and the on-off capacity for a class of sub-channel gain distributions.

Theorem 4 *If $E[\mu|\mu > x] - x$ is finite for all x, then*

$$C_N^{(\textit{wf})} \asymp C_N^{(\textit{on-off})} \asymp \mathcal{P}\mu_0^* \tag{7}$$

where μ_0^* is the optimal threshold and satisfies

$$\frac{\mathcal{P}}{N\bar{F}_\mu(\mu_0^*)} E^2[\mu|\mu > \mu_0^*] = 2\left(E[\mu|\mu > \mu_0^*] - \mu_0^*\right) \tag{8}$$

That is, the on-off power allocation is asymptotically optimal in the sense that the scaling with N is the same as with water pouring. To achieve this optimal scaling, it is necessary to feed back the allocation of *rates* across activated sub-channels. To limit the feedback with the on-off power allocation, we must therefore select the rate R_k for the k^{th} sub-channel from a discrete set $\mathcal{R} = \{0, \bar{R}_0, \cdots, \bar{R}_{n-1}\}$, where n is the number of of rate levels. That is, the range of channel gains is divided into intervals, $\{\nu_{n,i}\}_{i=0}^{n-1}$, each of which is associated with a particular rate $\bar{R}_i = \log\left(1 + \bar{P}\nu_{n,i}\right)$. Constant power $\bar{P}$ is assigned to sub-channels with $\mu_k \geq \nu_{n,0} = \mu_0$. If $\nu_{n,i} \leq \mu_k < \nu_{n,i+1}$ ($\nu_{n,n} = \infty$), then the corresponding rate $R_k = \bar{R}_i$, and $R_k = 0$ if $\mu_k < \mu_0$. The total achievable rate with this *finite-precision rate control* scheme is therefore $R^{(\text{fp})} = \max \sum_{k=1}^{N} R_k$.

Theorem 5 *If $E[\mu|\mu > x] - x$ is finite for all x, then given μ_0, the set of channel thresholds $\{\nu_{n,i}\}$, which maximizes $R^{(\textit{fp})}$ satisfies*

$$\begin{aligned} \nu_{n,0} &= \mu_0 \\ \nu_{n,i} &= \nu_{n,i-1} + \tfrac{\bar{F}(\mu_i) - \bar{F}(\mu_{i+1})}{f(\mu_i)} \qquad 1 \leq i \leq n-2 \\ \nu_{n,n-1} &= \nu_{n,n-2} + \tfrac{\bar{F}(\mu_{n-1})}{f(\mu_{n-1})} \end{aligned} \tag{9}$$

5.1 Rayleigh Fading

We now assume that $f_{\mu_i}(\mu) = e^{-\mu}$, corresponding to Rayleigh fading sub-channels. From (8), the optimal threshold $\mu_0^* \asymp \log N$, and Theorem 4 is restated as the following corollary.

Corollary 1 *With Rayleigh sub-channels, $C_N^{(\textit{wf})} \asymp C_N^{(\textit{on-off})} \asymp \mathcal{P}\log N$. Furthermore, the optimal number of active sub-channels $\bar{N}_a \asymp \mathcal{P}\log^2 N$.*

This implies that the feedback needed to specify all active subchannels grows as $\log^3 N$.

For comparison, we now consider the case in which only a finite number of sub-channels, M, can be activated. It is straightforward to show that asymptotically, the optimal scheme is to spread the power across the M sub-channels with the largest channel gains. We are interested in the asymptotic behavior of the corresponding channel capacity $C_N^{(M)}$, and the achievable rate with finite-precision rate control, $R^{(M)}$. In the latter case, with one bit to specify the rate for each sub-channel, the feedback $B_N = M \log N$.

Theorem 6 *With M active sub-channels $C_N^{(M)} \asymp M \log\log N$. Furthermore, if the feedback $B_N = M \log N$, then the achievable rate $R^{(M)}$ is a random variable with*

$$\begin{aligned} E[C_N^{(M)} - R^{(M)}] &\rightarrow (M-1)m_M - \textstyle\sum_{i=1}^{M-1} m_i \\ var[C_N^{(M)} - R^{(M)}] &\rightarrow \textstyle\sum_{i=1}^{M-1} \sigma_i^2 + (M-1)^2\sigma_M^2 \end{aligned} \tag{10}$$

where $m_i = E[\log Z]$, $\sigma_i^2 = var[\log Z]$, and Z is an order-i Gamma random variable.

The proof relies on the aymptotic theory of extreme statistics [9].

Given the threshold μ_0, the optimal channel thresholds for finite-precision rate control can be determined from Theorem 5. In this case, the increment $\nu_{n,i} - \nu_{n,i-1}$ depends only on n, and not on μ_0, and the achievable rate is given in the following theorem.

Theorem 7 *If $B_N \asymp \mathcal{P} \log^3 N$, then $R^{(fp)} \asymp C^{(on\text{-}off)}$. Furthermore, $C^{(on\text{-}off)} - R^{(fp)}$ converges in distribution to a Gaussian random variable with mean $\mathcal{P}(1 - e^{-(\nu_{n,1}-\mu_0)})$ and variance $2\mathcal{P}(1 - e^{-(\nu_{n,1}-\mu_0)})$.*

Figure 3 shows plots of mean data rate vs. SNR for multi-carrier transmission with 512 Rayleigh sub-channels. Achievable rates are shown with water-pouring ($C^{(\text{wf})}$), the optimal on-off power allocation with unlimited feedback, or infinite-precision rate control ($C^{(\text{on-off})}$), and on-off power allocation with finite-precision rate control with 1, 2, and 4 rate levels per active sub-channel. The figure shows that $C^{(\text{on-off})}$ is very close to $C^{(\text{wf})}$. As stated in Theorem 7, the gap between the achievable finite-precision data rate and $C^{(\text{wf})}$ increases with power. As expected, the gap between the finite- and infinite-precision rate curves decreases as n increases.

5.2 Correlated Fading

We now consider MC performance with limited feedback where the sub-channel sequence $\{\mu_i\}$ is a Markov chain. The mean capacity does not change, since it depends only on the first-order statistics; however, the correlation can be exploited to reduce the feedback. In what follows, we will assume finite-precision rate control with a given threshold μ_0. The feedback then represents a sequence of N indices, where each index indicates the rate assigned to the corresponding sub-channel. The minimum feedback is then the entropy of this sequence.

Let $\gamma = Pr\{\mu_i \geq \mu_0\}$ and $q = Pr\{\mu_i \geq \mu_0 | \mu_{i-1} \geq \mu_0\}$, i.e., q is the probability sub-channel i is active given sub-channel $i-1$ is active. Also, let $\bar{B}_N^{(\text{iid})}$ be the minimum feedback *per sub-channel* with i.i.d. sub-channels, in which case $q = \gamma$, and let $\bar{B}_N^{(\text{cor})}$ be the minimum feedback *per sub-channel* with correlated sub-channels. If $\frac{\gamma}{1-q} \rightarrow 0$ as $N \rightarrow \infty$, then $\frac{\bar{B}_N^{(\text{cor})}(\mu_0)}{\bar{B}_N^{(\text{iid})}(\mu_0)} \asymp 1 - q$.

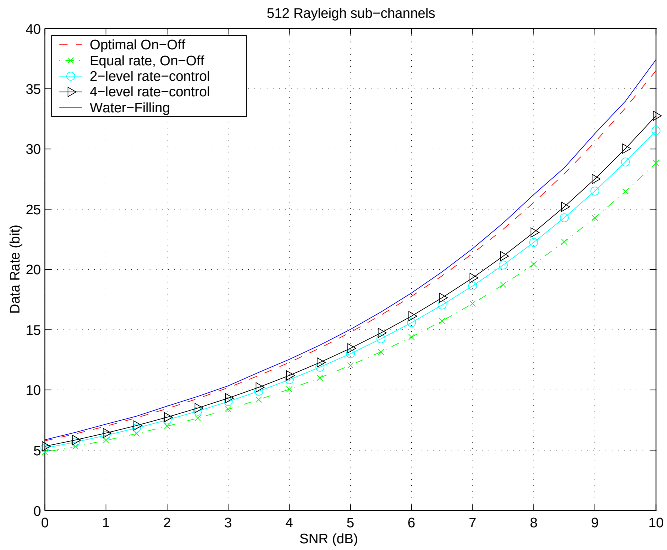

Figure 3: Mean channel capacity vs. SNR for water-filling and on-off power allocations with infinite- and finite-precision rate control.

As an example, consider correlated Rayleigh fading, where the sub-channel coefficient $h_i = \alpha h_{i-1} + \xi_i$ with $0 \leq \alpha \leq 1$, and ξ_i is a complex Gaussian random variable, which is independent of h_{i-1}. The parameter α determines the correlations between the sub-channels. The reduction in feedback obtained by exploiting the sub-channel correlations clearly depends on the rate at which $\alpha \to 1$ with N. Let W be the channel bandwidth, and define the coherence bandwidth, W_c, to be the bandwidth of L consecutive sub-channels with correlation larger than some given ρ, i.e., $L = \max\{l : \mathrm{cov}(\mu_i, \mu_{i+l}) \leq \rho\}$. The number of coherence bands spanned by the channel is assumed to be fixed, i.e., $\frac{W_c}{W} = L/N = \delta$ where δ is a constant. Applying this to the autoregressive model for $\{h_i\}$, we have $\alpha = e^{-\frac{\log \frac{1}{\rho}}{2\delta N}}$, and $\alpha \to 1$ as $N \to \infty$.

Corollary 2 *If* $\lim_{N\to\infty} \frac{\mu_0}{\frac{1}{2}\log N} \geq 1$, *then* $\bar{B}_N^{(iid)} \asymp e^{-\mu_0}\mu_0$, *and the feedback rate for the autoregressive channel model* $\bar{B}_N^{(cor)} \asymp \frac{\delta'}{\sqrt{N}} e^{-\mu_0}\mu_0^{\frac{3}{2}}$ *where* $\delta' = \sqrt{\frac{\log \frac{1}{\rho}}{\pi\delta}}$.

Comparing the feedback rates for the autoregressive and $i.i.d$ cases gives $\bar{B}_N^{(\mathrm{cor})}/\bar{B}_N^{(\mathrm{iid})} \asymp \delta' \sqrt{\frac{\mu_0}{N}}$. Specifically, if $\mu_0 \asymp b \log N$, where $b \geq 1/2$ is a constant, then $\bar{B}_N^{(\mathrm{cor})}/\bar{B}_N^{(\mathrm{iid})} \asymp \sqrt{b}\delta' \sqrt{\frac{\log N}{N}}$.

6 Conclusions

We have presented asymptotic performance results for wireless communications models with limited feedback. In the case of CDMA and MIMO channels, the feedback bits scale linearly with dimensions N and quadratically with N, respectively. The performance can be explicitly computed with RVQ, assuming a matched filter. For the MC model with Rayleigh fading sub-channels, a feedback rate, which scales as $O(\log^3 N)$ can achieve the maximum $O(\log N)$ asymptotic growth in capacity. This reduction in feedback is a result of the MC signatures being specified *a priori* as sinusoidal. Additional issues to be studied include the effect of inaccurate channel estimates on performance (i.e., the combination of limited and partial feedback), and determining the benefits of limited feedback strategies with time-varying channels.

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